Mini-superspace Quantization of the Reissner - Nordström geometry

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Outline

The mini-superspace system

Prototype mini-superspace Lagrangian

$$L = \frac{1}{2N(x)}\overline{G}_{\alpha\beta}(q(x))\dot{q}^{\alpha}\dot{q}^{\beta} - N(x)V(q(x)), \quad \alpha,\beta = 1,...,d = \dim \overline{G}_{\alpha\beta} \tag{1}$$

Spatially homogeneous spacetimes: $x \to t$
Static, spherically symmetric metric: $x \to r$

Constrained system of $d+1$ degrees of freedom $N(x)$, $q(x)$'s

Hamiltonian description

$$H_T = N\mathcal{H} + u_N p_N \tag{2}$$

where

$$p_N \approx 0 \tag{3a}$$

$$\mathcal{H} = \frac{1}{2}\overline{G}^{\alpha\beta}p_{\alpha}p_{\beta} + V(q) \approx 0 \tag{3b}$$

Assume a quantity $Q(x, q, p)$ at most linear in the momenta

$$\frac{d}{dx} Q \approx 0 \Rightarrow \frac{\partial Q}{\partial x} + \{Q, H_T\} = N \omega \mathcal{H} \approx 0. \tag{4}$$

For the system at hand

$$Q = \xi^\alpha(q) p_\alpha + \int N(x) \left[\omega(q(x)) + F(q(x))\right] V(q(x)) dx \tag{5}$$

with

$$\mathcal{L}_\xi \overline{G}_{\alpha\beta} = \omega(q) \overline{G}_{\alpha\beta}, \quad \text{and} \quad F(q) = \frac{1}{V(q)} \mathcal{L}_\xi V(q).$$

Whenever $\omega(q) = -F(q)$, we have an autonomous integral of motion

$$Q = \xi^\alpha(q) p_\alpha \tag{6}$$

with

$$\{Q, \mathcal{H}\} = \omega(q) \mathcal{H} \approx 0 \tag{7}$$

The constant potential parametrization

Reparameterize $N \mapsto n = N\,V$, then

$$\mathcal{H} = \frac{1}{2} G^{\alpha\beta} p_\alpha p_\beta + 1 \approx 0$$

with $G_{\alpha\beta} = V\,\overline{G}_{\alpha\beta}$ the scaled by the potential mini-superspace metric. Now

$$Q = \xi^\alpha(q) p_\alpha + \int N(x)\omega(q(x))dx \tag{8}$$

is an integral of motion whenever

$$\mathcal{L}_\xi G_{\alpha\beta} = \omega(q) G_{\alpha\beta}.$$

Killing vector fields ξ_I of $G_{\alpha\beta} \Rightarrow$ Autonomous $Q_I = \xi_I^\alpha p_\alpha$

$$\{Q_I, \mathcal{H}\} = 0$$

$$p_n \mapsto \widehat{p}_n = -\mathtt{i}\,\hbar \frac{\partial}{\partial n}, \quad p_\alpha \mapsto \widehat{p}_\alpha = -\mathtt{i}\,\hbar \frac{\partial}{\partial q^\alpha},$$

General linear Hermitian operator under measure $\mu = \sqrt{|\det G_{\mu\nu}|}$

$$\widehat{Q}_I = -\frac{\mathtt{i}\hbar}{2\mu}\left(\mu \xi_I^\alpha \partial_\alpha + \partial_\alpha(\mu \xi_I^\alpha)\right) = -\mathtt{i}\,\hbar\, \xi_I^\alpha \partial_\alpha \tag{9}$$

Hamiltonian operator

$$\widehat{\mathcal{H}} = -\frac{\hbar^2}{2\mu}\partial_\alpha \left(\mu G^{\alpha\beta}\partial_\beta\right) + \frac{d-2}{8(d-1)}\mathcal{R} + 1, \tag{10}$$

$$\{Q_I, Q_J\} = C_{IJ}^K Q_K \longmapsto [\widehat{Q}_I, \widehat{Q}_J] = -\mathtt{i}\,\hbar\, C_{IJ}^K \widehat{Q}_K,$$

Since ξ_I's are Killing vector fields of $G_{\alpha\beta}$

$$[\widehat{Q}_I, \widehat{\mathcal{H}}] = 0$$

Eigenvalue equations

$$\widehat{Q}_I \Psi = \kappa_I \Psi \tag{11}$$

The gravitational system

$$S = \frac{c^3}{16\pi G}\int\sqrt{-g}\,R\,d^4x - \frac{1}{4\mu_0}\int\sqrt{-g}F^{\mu\nu}F_{\mu\nu}d^4x \tag{12}$$

Field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{13a}$$

$$F^{\mu\nu}{}_{;\nu} = 0 \tag{13b}$$

Field strength and energy momentum tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}F^{\kappa\lambda}F_{\kappa\lambda}g_{\mu\nu}\right)$$

The static, spherically symmetric system

Line element

$$ds^2 = -b(r)^2 dt^2 + N(r)^2 dr^2 + a(r)^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \quad (14)$$

Electromagnetic potential

$$A = f(r) dt \quad (15)$$

Corresponding mini-superspace Lagrangian

$$L = \frac{2}{N(r)} \left(2a(r)\dot{a}(r)\dot{b}(r) + b(r)\dot{a}(r)^2 + \frac{a(r)^2 \dot{f}(r)^2}{b(r)} \right) + 2b(r)N(r) \quad (16)$$

Under the change $N(r) = \frac{n(r)}{2\,b(r)}$

$$L = \frac{4}{n(r)} \left(2a(r)b(r)\dot{a}(r)\dot{b}(r) + b(r)^2\dot{a}(r)^2 + a(r)^2\dot{f}(r)^2 \right) + n(r) \quad (17)$$

Mini-superspace metric

$$G_{\alpha\beta} = \begin{pmatrix} 8b^2 & 8ab & 0 \\ 8ab & 0 & 0 \\ 0 & 0 & 8a^2 \end{pmatrix}, \qquad (18)$$

Killing fields:

$$\xi_1 = \partial_f, \quad \xi_2 = \frac{1}{ab}\partial_b, \quad \xi_3 = \frac{f}{ab}\partial_b + \frac{1}{a}\partial_f, \quad \xi_4 = -a\partial_a + b\partial_b + f\partial_f$$
$$\xi_5 = \partial_a - \frac{b^2+f^2}{2ab}\partial_b - \frac{f}{a}\partial_f, \quad \xi_6 = af\partial_a - bf\partial_b - \frac{b^2+f^2}{2}\partial_f$$

and a homothecy $\xi_h = \frac{1}{4}\left(a\partial_a + b\partial_b + f\partial_f\right)$.
6 autonomous integrals of motion $Q_I = \xi_I^\alpha p_\alpha$ and a rheonomic

$$Q_h = \xi_h^\alpha p_\alpha - \int n(r)dr. \qquad (19)$$

Purely algebraic derivation of the solution without fixing the gauge:
Solve $Q_i = c_i$, $i = 1, ..., 5$ and $Q_h = 0$ with respect to $a, \dot{a}, b, \dot{b}, \int N dr, N$
The consistency condition

$$\frac{d}{dr} \int N(r) dr = N(r)$$

leads to the quadratic constraint equation: $c_5 = \frac{16 - c_3^2}{2c_2}$
By the re-parametrizations $c_1 = 4Q$, $c_2 = 4/\tilde{c}$, $c_4 = \tilde{c}\,(c_3 Q - 4M)$

$$\begin{aligned} ds^2 = & -\tilde{c}^2 \left(1 - \frac{2M}{a} + \frac{Q^2}{a^2}\right) dt^2 + \left(1 - \frac{2M}{a} + \frac{Q^2}{a^2}\right)^{-1} da^2 \\ & + a^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \quad (20)$$

$$A = \tilde{c} \left(\frac{c_3}{4} - \frac{Q}{a(r)}\right) dt. \quad (21)$$

Quantization on the flat mini-superspace

Transformation $(a, b, f) \mapsto (\chi, \psi, \zeta)$, where $G_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$

$$\begin{gathered} X_1 = \partial_\chi, \quad X_2 = \partial_\phi, \quad X_3 = \partial_\zeta \\ X_4 = \zeta\partial_\chi + \chi\partial_\zeta, \quad X_5 = \psi\partial_\chi + \chi\partial_\psi, \quad X_6 = \psi\partial_\zeta - \zeta\partial_\psi. \end{gathered} \tag{22}$$

The constant values that the conserved quantities $\tilde{Q}_I = X_I^\alpha p_\alpha$

$$\begin{aligned} &\tilde{Q}_1 = \frac{\tilde{c}(16 - c_3^2)}{64} - \frac{2}{\tilde{c}}, \quad \tilde{Q}_2 = \frac{c_3}{2\sqrt{2}}, \quad \tilde{Q}_3 = \frac{\tilde{c}(c_3^2 - 16)}{64} - \frac{2}{\tilde{c}}, \\ &\tilde{Q}_4 = \tilde{c}\,(c_3 Q - 4M), \quad \tilde{Q}_5 = \frac{Q\left(128 - \tilde{c}^2(16 + c_3^2)\right) + 8\,\tilde{c}^2\, c_3\, M}{16\sqrt{2}}, \\ &\tilde{Q}_6 = \frac{Q\left(128 + \tilde{c}^2(16 + c_3^2)\right) - 8\,\tilde{c}^2\, c_3\, M}{16\sqrt{2}} \end{aligned} \tag{23}$$

Casimir invariant of the semi-simple subalgebra spanned by X_4, X_5 and X_6

$$Q_{Cas} = X_6 \otimes X_6 - X_4 \otimes X_4 - X_5 \otimes X_5 \tag{24}$$

which on mass shell assumes the value

$$\tilde{Q}_{Cas} = \tilde{Q}_6^2 - \tilde{Q}_4^2 - \tilde{Q}_5^2 = 16\tilde{c}^2(Q^2 - M^2). \tag{25}$$

Quantization with respect to X_6

Transformation $(\chi, \psi, \zeta) \mapsto (u, v, w)$ to bring X_6 into normal form.
Operators

$$\widehat{Q}_6 = -\mathrm{i}\frac{\partial}{\partial w} \tag{26}$$

$$\widehat{Q}_{Cas} = \widehat{Q}_6^2 - \widehat{Q}_4^2 - \widehat{Q}_5^2 = \frac{1}{\cosh \mathrm{v}}\frac{\partial}{\partial \mathrm{v}}\left(\cosh \mathrm{v}\frac{\partial}{\partial \mathrm{v}}\right) - \frac{1}{\cosh^2 \mathrm{v}}\frac{\partial^2}{\partial w^2}, \tag{27}$$

$$\widehat{H} = -\frac{1}{2u^2}\left[\frac{\partial}{\partial u}\left(u^2\frac{\partial}{\partial u}\right) - \frac{1}{\cosh \mathrm{v}}\frac{\partial}{\partial \mathrm{v}}\left(\cosh \mathrm{v}\frac{\partial}{\partial \mathrm{v}}\right) + \frac{1}{\cosh^2 \mathrm{v}}\frac{\partial^2}{\partial w}\right] - 1. \tag{28}$$

Eigenvalue equations

$$\widehat{Q}_6\Psi_{k\ell}(u, v, w) = k\Psi_{k\ell}(u, v, w) \tag{29a}$$

$$\widehat{Q}_{Cas}\Psi_{k\ell}(u, v, w) = \ell(\ell + 1)\Psi_{k\ell}(u, v, w), \tag{29b}$$

$$\widehat{H}\Psi_{k\ell}(u, v, w) = 0 \tag{29c}$$

Solution of the form $\Psi_{k\ell} = \psi^{(1)}_{\ell}(u)\psi^{(2)}_{k\ell}(\mathrm{v})\psi^{(3)}_{k}(w)$.
Due to the boundary condition on the rotation X_6, $\psi^{(3)}(0) = \psi^{(3)}(2\pi)$

$$\psi^{(3)}_k(w) = \frac{1}{\sqrt{2\pi}} e^{\mathbb{i}\, k\, w}, \quad k \in \mathbb{Z} \tag{30}$$

Equation for the Casimir invariant

$$\frac{1}{\cosh \mathrm{v}} \frac{d}{d\mathrm{v}} \left(\cosh \mathrm{v} \, \frac{d\psi^{(2)}_{k\ell}(\mathrm{v})}{d\mathrm{v}} \right) - \left[\ell(\ell+1) - \frac{k^2}{\cosh^2 \mathrm{v}} \right] \psi^{(2)}_{k\ell}(\mathrm{v}) = 0. \tag{31}$$

Under the transformation $\psi^{(2)}_{k\ell}(\mathrm{v}) = \frac{\Phi_{k\ell}(\mathrm{v})}{\cosh^{1/2} \mathrm{v}}$

$$\frac{d^2 \Phi_{k\ell}(\mathrm{v})}{d\mathrm{v}^2} + \left[\frac{k^2 - \frac{1}{4}}{\cosh^2 \mathrm{v}} - \frac{1}{4}(2\ell+1)^2 \right] \Phi_{k\ell}(\mathrm{v}) = 0 \tag{32}$$

with

$$\frac{1}{4} - k^2 = -V_0 < 0, \quad E = -\frac{1}{4}(2\ell+1)^2 \tag{33}$$

- Case $\ell(\ell+1) \geq 0$ (classically $Q_{Cas} \geq 0 \Rightarrow Q \geq M$): The spectrum is discrete with the quantum condition

$$k > \ell, \quad k \in \mathbb{Z}_+, \quad \ell \in \mathbb{N} \tag{34}$$

$$\psi^{(2)}_{k\ell}(\mathrm{v}) \propto \frac{{}_2F_1(\ell - k + 1, \ell + k + 1; \ell + \frac{3}{2}; \frac{1}{2}(1 - \tanh \mathrm{v}))}{\cosh^{\ell - 1/2} \mathrm{v}} \tag{35}$$

- Case $\ell(\ell+1) < 0$ (classically $Q < M$): $\ell = \mathrm{i}\, s - \frac{1}{2}$ and the spectrum is continuous, $s \in \mathbb{R}$.

$$\psi^{(2)}_{k\ell}(\mathrm{v}) \propto P^{k}_{\mathrm{i}\, s - \frac{1}{2}}(\mathrm{i}\, \sinh \mathrm{v}) \tag{36}$$

$$\int_{-\infty}^{+\infty} P^{k}_{\mathrm{i}s-1/2}(\mathrm{i}\, x)(P^{k}_{\mathrm{i}p-1/2}(\mathrm{i}\, x))^{*} dx = A(p,s)\delta(p-s) + A(p,-s)\delta(p+s)$$

$$A(p,s) = \cosh[(p+s)\frac{\pi}{2}]\left[\frac{2^{-\mathrm{i}(p-s)}\Gamma(-\mathrm{i}p)\Gamma(\mathrm{i}s)}{\Gamma(\frac{1}{2} - k - \mathrm{i}p)\Gamma(\frac{1}{2} - k + \mathrm{i}s)} + (s \leftrightarrow p)\right] \tag{37}$$

The WDW equation leads to:

$$\psi_\ell^{(1)}(u) \propto j_\ell(\sqrt{2}u) \tag{38}$$

Orthonormality condition for spherical Bessel $j_\ell(u)$

$$\int_0^{+\infty} u^2 j_\ell(\alpha_1 u) j_\ell(\alpha_2 u) du = \frac{\pi}{2\alpha_1^2}\delta(\alpha_1 - \alpha_2),$$

Symbolically we write

$$\int_0^{+\infty} u^2 \psi_\ell^{(1)}(u)^* \psi_\ell^{(1)}(u) du \propto \int_0^{+\infty} u^2 j_\ell(\sqrt{2}u) j_\ell(\sqrt{2}u) = \delta(0), \tag{39}$$

to which

$$\rho(u, \mathrm{v}, w) = \frac{\mu \Psi_{k\ell}^* \Psi_{k\ell}}{\delta(0)} \tag{40}$$

is normalized.

Normalized wave functions

- Case $\ell(\ell+1) \geq 0$, $k > \ell$, $k \in \mathbb{Z}_+$, $\ell \in \mathbb{N}$

$$\Psi_{k\ell} = C_1 j_\ell(\sqrt{2}u)\, \frac{{}_2F_1(\ell-k+1, \ell+k+1; \ell+\frac{3}{2}; \frac{1}{2}(1-\tanh \mathrm{v}))}{\cosh^{\ell-1/2} \mathrm{v}} e^{\mathrm{i}kw}$$

$$C_1 = \left(\frac{2\Gamma(k+\ell+1)\Gamma(\ell+1)}{\pi^{5/2}(k-\ell-1)!\Gamma(2\ell+2)\Gamma(\ell+\frac{1}{2})} \right)^{1/2}$$

- Case $\ell(\ell+1) < 0$, $k \in \mathbb{Z}$, $s \in \mathbb{R}$

$$\Psi_{ks} = C_2\, j_{\mathrm{i}s-\frac{1}{2}}(\sqrt{2}u)\, P^k_{\mathrm{i}s-\frac{1}{2}}(\mathrm{i}\sinh \mathrm{v})\, e^{\mathrm{i}kw}$$

$$C_2 = \frac{1}{\pi} \left(\frac{\Gamma(\frac{1}{2}-k-\mathrm{i}s)\Gamma(\frac{1}{2}-k+\mathrm{i}s)}{\cosh(s\pi)\Gamma(-\mathrm{i}s)\Gamma(\mathrm{i}s)} \right)^{1/2}$$

Relations for M and Q

For both cases ($c_3 = 0$)

$$M = \frac{\sqrt{32\tilde{c}^2 k^2 - (\tilde{c}^2+8)^2 \ell(\ell+1)}}{4\,|\tilde{c}|(\tilde{c}^2+8)} \tag{41a}$$

$$Q = \frac{\sqrt{2}\,k}{(\tilde{c}^2+8)}. \tag{41b}$$

If we set $\tilde{c} = 2\sqrt{2}$

$$\begin{aligned} &\tilde{Q}_1 = 0, \quad \tilde{Q}_2 = \frac{1}{2}, \quad \tilde{Q}_3 = -\sqrt{2} \\ &\tilde{Q}_4 = -8\sqrt{2}M, \quad \tilde{Q}_5 = 0, \quad \tilde{Q}_6 = 8\sqrt{2}Q. \end{aligned} \tag{42}$$

$$M = \frac{1}{8\sqrt{2}}\sqrt{k^2 - \ell(\ell+1)} \tag{43a}$$

$$Q = \frac{k}{8\sqrt{2}}. \tag{43b}$$

Actual mass and charge

$$M = \frac{G\,m}{c^2}, \quad Q = \frac{q}{c^2}\sqrt{\frac{G}{4\pi\varepsilon_0}}. \tag{44}$$

Introduce a constant d with units of distance in place of $\hbar$.

$$M = \frac{1}{8\sqrt{2}}\sqrt{k^2 - \ell(\ell+1)}d \tag{45a}$$

$$Q = \frac{1}{8\sqrt{2}}k\,d. \tag{45b}$$

For the base state $k = 1$, $\ell = 0$

$$d = 4\sqrt{2}\frac{q}{c^2}\sqrt{\frac{G}{\pi\varepsilon_0}}. \tag{46}$$

If we consider the fundamental charge $q = |e| \simeq 1.602 \cdot 10^{-19}\mathrm{C}$

$$d \simeq 1.562 \cdot 10^{-35}\mathrm{m} \sim \ell_P \simeq 1.616 \cdot 10^{-35}\mathrm{m}$$

Conclusions

- Reduction to the mini-superspace system
- Use of the conditional symmetries of the constrained system to define eigenequations.
- Classically the solution can be derived purely algebraically.
- In quantization:
  - Discrete spectrum when $Q \geq M$
  - Continuous spectrum if $Q < M$
  - Introduction of a fundamental distance to which Q, M are quantized, $d \sim \ell_P$