# Conformal Anomalies and Einstein Equations

***Hermann Nicolai***
***MPI für Gravitationsphysik, Potsdam***
***(Albert Einstein Institut)***

**Work based on:**

*K. Meissner and H.N.:* `arXiv:1607.07312`

*H. Godazgar, K. Meissner and H.N.:* `arXiv:1612.01296`

*and work in progress*

## Executive Summary

**Is the cancellation of conformal anomalies required**

- **Quantum mechanically: to ensure quantum consistency of perturbative quantum gravity?**

**... in analogy with cancellation of gauge anomalies for Standard Model (where cancellation is required to maintain renormalizability), and/or**

- **already at classical level: corrections from induced anomalous non-local action to Einstein Field Equations may potentially overwhelm smallness of Planck scale $\ell_{\rm PL}$ $\Rightarrow$ huge corrections to any solution?**

**If so, cancellation requirement could lead to *very strong* restrictions on admissible theories!**

*See also: G. 't Hooft, Int.J.Mod.Phys. D24(2015)154001*

# Conformal Symmetry

**Conformal symmetry comes in two versions:**

1. **Global conformal symmetry = extension of Poincaré group by dilatations $D$ and conformal boosts $K^\mu$**

2. **Local dilatations = Weyl transformations**

$$g_{\mu\nu}(x) \;\to\; e^{2\sigma(x)} g_{\mu\nu}(x)$$

**Important consequence: flat space limit of Weyl and diffeomorphism invariant theories exhibits full (global) conformal symmetry (via conformal Killing vectors)**

$\to$ **important restrictions on effective actions $\Gamma = \Gamma[g]$ with $\Gamma[g]$ as the generating functional for correlators of energy momentum tensor $\langle T_{\mu_1\nu_1}(x_1)\cdots T_{\mu_n\nu_n}(x_n)\rangle\Big|_{g_{\mu\nu}(x)=\eta_{\mu\nu}}$.**

# Conformal Anomaly $\equiv$ Trace Anomaly

**Conformal anomaly ($\equiv$ trace anomaly)** [Deser,Duff,Isham(1976)]

$$T_\mu{}^\mu(x) = a\,\mathbf{E}_2(x) \equiv a R(x) \qquad (D=2)$$

$$T_\mu{}^\mu(x) = \mathcal{A}(x) \equiv a\,\mathbf{E}_4(x) + c\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}(x) \quad (D=4)$$

**where $\mathbf{E}_4(x) \equiv$ Euler number density**

$$\mathbf{E}_4 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

**Coefficients $c_s$ and $a_s$ for fields of spin $s$ (with $s \leq 2$) were computed already long ago.**

[Duff(1977);Christenses,Duff(1978);Fradkin,Tseytlin(1982); Tseytlin(2013);Eguchi,Gilkey, Hanson, Phys.Rep.66(1980)213; see also *"Path integrals and anomalies in curved space"* by Bastianelli,van Nieuwenhuizen]

**NB: further contribution to anomaly $\propto \Box R$ can be removed by *local* counterterm $R^2$.**

## Anomalous Effective Action

**Anomaly can be obtained by varying anomalous effective action $\Gamma_{\rm anom} = \Gamma_{\rm anom}[g]$**

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}}g_{\mu\nu}(x)\frac{\delta\Gamma_{\rm anom}[g]}{\delta g_{\mu\nu}(x)}$$

**but this effective action is necessarily *non-local.***

**Simplest example: string theory in *non-critical* dimension has a trace anomaly $T^{\mu}{}_{\mu} \propto R$ $\Rightarrow$ leads to anomalous effective action = Liouville theory.** [Polyakov(1981)]

$$\Gamma_{\rm anom}^{D=2} \propto \int d^2x\sqrt{-g}R\,\Box_g^{-1}R$$

- **new propagating degree of freedom (longitudinal mode of world sheet metric = Liouville field)**

  $\Rightarrow$ **changes physics in dramatic ways!**

**Analog for gravity in $D = 4$: non-local actions that give $a$-anomaly *exactly* are known, for instance** [Riegert(1984)]

$$\Gamma_{\text{anom}}[g] = \int d^4x d^4y \sqrt{-g(x)}\sqrt{-g(y)} \left(\mathbf{E}_4 - \frac{2}{3}\Box_g R\right)(x) G^P(x,y) \left(\mathbf{E}_4 - \frac{2}{3}\Box_g R\right)(y)$$

**with $\triangle^P G^P(x) = \delta^{(4)}(x)$, and 4th order operator** [Paneitz(1983)]

$$\triangle^P \equiv \Box_g \Box_g + 2\nabla_\mu \left(R^{\mu\nu} - \frac{1}{3} g^{\mu\nu} R\right) \nabla_\nu$$

**However, no closed form actions are known that have the correct conformal properties (as would be obtained from Feynman diagrams), despite many efforts.**

[Deser,Schwimmer(1993);Erdmenger,Osborn(1998);Deser(2000);Barvinsky et al.(1998); Mazur,Mottola(2001);...]

**In lowest order**

$$\Gamma^{D=4}_{\text{anom}} \propto \int d^4x \sqrt{-g}\mathbf{E}_4 \Box_g^{-1} R + \cdots$$

**where $\cdots$ stands for *infinitely many* (non-local) terms.**

**While**

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}} g_{\mu\nu}(x) \frac{\delta\Gamma_{\rm anom}[g]}{\delta g_{\mu\nu}(x)}$$

***is*** **local, contribution to Einstein equations**

$$\ell_{PL}^{-2}\Big[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\Big] = -\frac{2}{\sqrt{-g(x)}} \frac{\delta\Gamma_{\rm anom}[g]}{\delta g^{\mu\nu}(x)} + \cdots$$

**in general *remains non-local* for non-scalar modes.**

**Claim: non-localities from $\Box_g^{-1}$ in $\Gamma_{\rm anom}[g]$ can 'overwhelm' smallness of Planck scale and produce observable deviations for Einstein's equations!**

**... if $\Gamma_{\rm anom}$ is to be added to classical action** [cf.E.Mottola (since 2001)]

**Typical correction is (symmetrized traceless part of)**

$$\propto \nabla_\mu\big(G^{\rm ret} \star \mathbf{E}_4\big)\, \nabla_\nu\big(G^{\rm ret} \star R\big) + \cdots$$

**with retarded propagator $G^{\rm ret}$ in space-time background given by metric $g_{\mu\nu}$ solving classical Einstein equations.**

**For order of magnitude estimate, evaluate this integral for a (conformally flat) cosmological background**

$$ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2)$$

**by integrating from end of radiation era** ($= t_{\rm rad}$) **back to** $t_0 = n_* \ell_{\rm PL}$**, with** $a(\eta) = \eta/(2t_{\rm rad})$ **and** $\eta = 2\sqrt{t t_{\rm rad}}$ **and with retarded Green's function** [Waylen(1978)]

$$G^{\rm ret}(\eta, \mathbf{x}; \eta', \mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \cdot \frac{\delta(\eta - \eta' - |\mathbf{x}-\mathbf{y}|)}{a(\eta)a(\eta')}$$

**Resulting correction on r.h.s. of Einstein's equations**

$$T_{00}^{\rm anom} \sim 10^{-5}\, t_{\rm rad}^{-1}\, (n_* \ell_{\rm PL})^{-3}$$

**'beats' factor** $\sim (t_{\rm rad}\ell_{\rm PL})^{-2}$ **on l.h.s. even for** $n_* \sim 10^8$ **!**

**Similar results from evaluating contribution of Riegert action with Green's function** $\triangle^P G_P(x,y) = \delta^{(4)}(x,y)$

$$G_P(\eta, \mathbf{x}; \eta', \mathbf{y}) = \frac{1}{8\pi}\theta(\eta - \eta' - |\mathbf{x}-\mathbf{y}|) \quad \textit{(for any } a(\eta)\textit{!)}$$

# Modifications of Einstein's equations

→ corrections can be *exactly* evaluated for conformally flat background [Goadazgar,Meissner,HN:1612.01296]

$$\ell_{PL}^{-2} G_{\mu\nu} \propto \int d^4y \int d^4z \sqrt{-g(x)} \nabla_\alpha \nabla_\mu \nabla_\nu G_P(x,y) \nabla^\alpha G_P(x,z)$$

$$+ \textit{plus many more terms}$$

Evaluation of integrals for conformally flat background produces many terms of the same order of magnitude as l.h.s. ... idem for slightly non-homogeneous backgrounds, and for action with dilaton $\tau$ and spontaneously broken conformal symmetry [Schwimmer,Theisen(2011)]

$$W = -a \int d^4x \sqrt{-g} \left( \frac{1}{f} \tau \mathbf{E}_4 + \frac{2}{f^2} G^{\mu\nu} \partial_\mu \tau \partial_\nu \tau + \frac{4}{f^3} \partial^\mu \tau \partial_\mu \tau \Box \tau - \frac{2}{f^4} (\partial^\mu \tau \partial_\mu \tau)^2 \right)$$

→ looks like generic phenomenon, and could thus affect *any* solution of Einstein equations!

⇒ need to cancel conformal anomalies?

# Cancelling conformal anomalies

| | massless | | massive | |
|---|---|---|---|---|
| | $c_s$ | $a_s$ | $\bar{c}_s$ | $\bar{a}_s$ |
| $0(0^*)$ | $\frac{3}{2}(\frac{3}{2})$ | $-\frac{1}{2}(\frac{179}{2})$ | $\frac{3}{2}(\varnothing)$ | $-\frac{1}{2}(\varnothing)$ |
| $\frac{1}{2}$ | $\frac{9}{2}$ | $-\frac{11}{4}$ | $\frac{9}{2}$ | $-\frac{11}{4}$ |
| $1$ | $18$ | $-31$ | $\frac{39}{2}$ | $-\frac{63}{2}$ |
| $\frac{3}{2}$ | $-\frac{411}{2}$ | $\frac{589}{4}$ | $-201$ | $\frac{289}{2}$ |
| $2$ | $783$ | $-571$ | $\frac{1605}{2}$ | $-\frac{1205}{2}$ |

- $\bar{c}_s$ **and** $\bar{a}_s$ **include lower helicities:** $\bar{c}_1 = c_1 + c_0$, ***etc.*** **(but numbers need to be re-checked for** $s = \frac{3}{2}, 2$ **!)**
- **Gravitinos and supergravity needed for cancellation**
- **No cancellation possible for** $N \leq 4$ **supergravities**

**NB: gravitino contribution may evade positivity properties because of ghost contribution *and* because there does not exist a gauge invariant traceless energy momentum tensor for $s = \frac{3}{2}$.**

$$c_2 + 5c_{\frac{3}{2}} + 10c_1 + 11c_{\frac{1}{2}} + 10c_0 = 0 \qquad (N = 5)$$
$$c_2 + 6c_{\frac{3}{2}} + 16c_1 + 26c_{\frac{1}{2}} + 30c_0 = 0 \qquad (N = 6)$$
$$c_2 + 8c_{\frac{3}{2}} + 28c_1 + 56c_{\frac{1}{2}} + 70c_0 = 0 \qquad (N = 8)$$

**Old result: combined contribution $\sum_s (c_s + a_s)$ vanishes for all $N \geq 3$ theories with appropriate choice of field representations for spin zero fields** [Townsend,HN(1981)]**.**

**Thus: conformal anomalies for $\sum_s a_s$ and $\sum_s c_s$ cancel only for $N \geq 5$ supergravities!** [Meissner,HN(2016)]

**... as they do for 'composite' U(5), U(6) and SU(8) R-symmetry anomalies.** [Marcus(1985)]

**Related to possible finiteness of $N \geq 5$ supergravities?**

## Idem for D=11 SUGRA compactified $AdS_4 \times S^7$

| | $SO(8)$ **representations** |
|---|---|
| 0 | $[n{+}2\ 0\ 0\ 0]$, $[n\ 0\ 2\ 0]$, $[n{-}2\ 2\ 0\ 0]$, $[n{-}2\ 0\ 0\ 2]$, $[n{-}2\ 0\ 0\ 0]$ |
| $\frac{1}{2}$ | $[n{+}1\ 0\ 1\ 0]$, $n{-}1\ 1\ 1\ 0]$, $[n{-}2\ 1\ 0\ 1]$, $[n{-}2\ 0\ 0\ 1]$ |
| 1 | $[n\ 1\ 0\ 0]$, $[n{-}1\ 0\ 1\ 1]$ , $[n{-}2\ 1\ 0\ 0]$ |
| $\frac{3}{2}$ | $[n\ 0\ 0\ 1]$ , $[n{-}1\ 0\ 1\ 0]$ |
| 2 | $[n\ 0\ 0\ 0]$ |

**'Floor-by-floor' cancellation** [Cf.Gibbons,HN(1985)]**: for all** $n$

$$\bar{c}_2 f_2(n) + \bar{c}_{\frac{3}{2}} f_{\frac{3}{2}}(n) + \bar{c}_1 f_1(n) + \bar{c}_{\frac{1}{2}} f_{\frac{1}{2}}(n) + \bar{c}_0 f_0(n) = 0$$

**where** $f_s(n) \equiv \sum$ **(dimensions of SO(8) spin-**$s$ **irreps) at Kaluza-Klein level** $n$ **(no anomalies for odd** $D$**).**

## Conceptual Issues

**Why worry about conformal anomalies in theories that are not even classically conformally invariant?**

**HOWEVER: recall axial anomaly and anomalous conservation of axial current**

$$\partial^\mu J_\mu^5 = 2im\bar{\psi}\gamma^5\psi + \frac{\alpha}{8\pi}F^{\mu\nu}\tilde{F}_{\mu\nu}$$

**$\rightarrow$ anomaly is crucial *even in presence of explicitly broken axial symmetry* ($m \neq 0$).**

**Idem for gauge anomalies in Standard Model: these must cancel even when quarks and leptons acquire masses via spontaneous symmetry breaking.**

**Is there a hidden conformal structure behind $N \geq 5$ supergravities (and M Theory)? But cannot be conformal supergravity in any conventional sense...**

# Outlook

***V. Mukhanov: "You cannot figure out the fundamental theory by simply looking at the sky!"***

*But maybe there is a way...*