Applications of comparison geometry a la Bakry-Émery to static and stationary solutions.

Martin Reiris.

Centro de Matemática
Facultad de Ciencias, Uruguay.

March 31, 2017.

What is comparison geometry?

A set of techniques allowing to obtain inequalities of geometric quantities, like distances and angles, on a manifold (M, g) that enjoys a curvature condition, like $Ric \geq kg$, k constant.

What is comparison geometry?

A set of techniques allowing to obtain inequalities of geometric quantities, like distances and angles, on a manifold (M, g) that enjoys a curvature condition, like $Ric \geq kg$, k constant.

Example 1 (E. Calabi 1960)
On $(M; g)$ suppose that $Ric \geq 0$ and let $d(p) = dist(p, o)$, o fixed point. Then,

$$\Delta d \leq \frac{n-1}{d} \tag{1}$$

What is comparison geometry?

A set of techniques allowing to obtain inequalities of geometric quantities, like distances and angles, on a manifold (M, g) that enjoys a curvature condition, like $Ric \geq kg$, k constant.

Example 1 (E. Calabi 1960)
On $(M; g)$ suppose that $Ric \geq 0$ and let $d(p) = dist(p, o)$, o fixed point. Then,

$$\Delta d \leq \frac{n-1}{d} \tag{1}$$

Note that on the Euclidean $\mathbb{R}^n$ where $Ric = 0$ we have

$$\Delta d = \frac{n-1}{d} \tag{2}$$

(To the purpose of this talk) Comparison geometry a la Bakry-Émery, can be defined similarly but using Ric_f instead of Ric and Δ_f instead of Δ, where

$$Ric_f^{\alpha} := Ric + \nabla\nabla f - \alpha \nabla f \nabla f, \tag{3}$$

$$\Delta_f \varphi := \Delta\varphi + \langle \nabla f, \nabla \varphi \rangle, \tag{4}$$

$\alpha \geq 0$ is a constant and f is an arbitrary function

(To the purpose of this talk) Comparison geometry a la Bakry-Émery, can be defined similarly but using Ric_f instead of Ric and Δ_f instead of Δ, where

$$Ric_f^{\alpha} := Ric + \nabla\nabla f - \alpha \nabla f \nabla f, \tag{3}$$

$$\Delta_f \varphi := \Delta \varphi + \langle \nabla f, \nabla \varphi \rangle, \tag{4}$$

$\alpha \geq 0$ is a constant and f is an arbitrary function

In these new comparison techniques there are no constraints on $\alpha \geq 0$ and f, this is what makes the techniques so powerful.

(To the purpose of this talk) Comparison geometry a la Bakry-Émery, can be defined similarly but using Ric_f instead of Ric and Δ_f instead of Δ, where

$$Ric_f^\alpha := Ric + \nabla\nabla f - \alpha \nabla f \nabla f, \tag{3}$$

$$\Delta_f \varphi := \Delta\varphi + \langle \nabla f, \nabla \varphi \rangle, \tag{4}$$

$\alpha \geq 0$ is a constant and f is an arbitrary function

In these new comparison techniques there are no constraints on $\alpha \geq 0$ and f, this is what makes the techniques so powerful.

Example 2 (J. Case 2010)
On $(M; g)$, with M non-compact, suppose that,

$$Ric_f^\alpha \geq 0, \qquad \text{and} \qquad \Delta_f \varphi \geq c\varphi^2 \tag{5}$$

for some $\alpha > 0$, f and function $\varphi \geq 0$.

(To the purpose of this talk) Comparison geometry a la Bakry-Émery, can be defined similarly but using Ric_f instead of Ric and Δ_f instead of Δ, where

$$Ric_f^{\alpha} := Ric + \nabla\nabla f - \alpha \nabla f \nabla f, \tag{3}$$

$$\Delta_f \varphi := \Delta\varphi + \langle \nabla f, \nabla\varphi \rangle, \tag{4}$$

$\alpha \geq 0$ is a constant and f is an arbitrary function

In these new comparison techniques there are no constraints on $\alpha \geq 0$ and f, this is what makes the techniques so powerful.

Example 2 (J. Case 2010)
On $(M; g)$, with M non-compact, suppose that,

$$Ric_f^{\alpha} \geq 0, \qquad \text{and} \qquad \Delta_f \varphi \geq c\varphi^2 \tag{5}$$

for some $\alpha > 0, f$ and function $\varphi \geq 0$. Then,

$$\varphi(p) \leq \frac{c}{d^2(p, \partial M)} \tag{6}$$

What all this has to do with General Relativity?

What all this has to do with General Relativity?

The point is that certain equations related to static and stationary solutions in GR can be often casted in the form of **Example 2**.

What all this has to do with General Relativity?

The point is that certain equations related to static and stationary solutions in GR can be often casted in the form of **Example 2**.

For instance for,

1. Vaccum static solutions
2. Einstein/Klein-Gordon static solutions
3. Axisymmetric vaccum stationary solutions

Vaccum static solutions.

The static vaccum equations are,

$$N Ric = \nabla\nabla N, \qquad \text{and} \qquad \Delta N = 0 \tag{7}$$

Making $f = \ln N$ they become,

$$Ric^1_f = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{8}$$

Vaccum static solutions.

The static vaccum equations are,

$$NRic = \nabla\nabla N, \qquad \text{and} \qquad \Delta N = 0 \tag{7}$$

Making $f = \ln N$ they become,

$$Ric^1_f = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{8}$$

These two equations can be used in a general Bochner formula to give,

$$\Delta\varphi \geq 2\varphi^2, \qquad \text{where} \qquad \varphi = |\nabla f|^2. \tag{9}$$

Vaccum static solutions.

The static vaccum equations are,

$$NRic = \nabla\nabla N, \qquad \text{and} \qquad \Delta N = 0 \tag{7}$$

Making $f = \ln N$ they become,

$$Ric^1_f = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{8}$$

These two equations can be used in a general Bochner formula to give,

$$\Delta\varphi \geq 2\varphi^2, \qquad \text{where} \qquad \varphi = |\nabla f|^2. \tag{9}$$

Hence, if $\partial M = 0$ and M is non-compact, the **Example 2** shows that $\varphi(p) \leq c/d(p, \infty) = 0$, thus $f = const$ and therefore $Ric = 0$.

Vaccum static solutions.

The static vaccum equations are,

$$NRic = \nabla\nabla N, \qquad \text{and} \qquad \Delta N = 0 \tag{7}$$

Making $f = \ln N$ they become,

$$Ric^1_f = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{8}$$

These two equations can be used in a general Bochner formula to give,

$$\Delta\varphi \geq 2\varphi^2, \qquad \text{where} \qquad \varphi = |\nabla f|^2. \tag{9}$$

Hence, if $\partial M = 0$ and M is non-compact, the **Example 2** shows that $\varphi(p) \leq c/d(p,\infty) = 0$, thus $f = const$ and therefore $Ric = 0$.
Thus, (for static solutions with no singularity, no matter equals no gravity)

Theorem

(M. T. Anderson, 2000) The only geodesically complete vaccum static solutions are either Minkowski or a quotient thereof.

Recall the static vaccum equations are,

$$Ric_f^1 = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{10}$$

Recall the static vaccum equations are,

$$Ric_f^1 = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{10}$$

The form of these equation is preserved under conformal transformations. Fix ϵ and define $\overline{g} = N^{-2\epsilon} g$. Then,

$$\overline{Ric}_f^{\alpha} = 0, \qquad \text{and} \qquad \overline{\Delta}_f f = 0, \tag{11}$$

where $\alpha = (1 - 2\epsilon - \epsilon^2)/(1+\epsilon)^2$ and $f = -(1+\epsilon)\ln N$.

Recall the static vaccum equations are,

$$Ric_f^1 = 0, \qquad \text{and} \qquad \Delta_f f = 0. \tag{10}$$

The form of these equation is preserved under conformal transformations. Fix ϵ and define $\overline{g} = N^{-2\epsilon} g$. Then,

$$\overline{Ric}_f^{\alpha} = 0, \qquad \text{and} \qquad \overline{\Delta}_f f = 0, \tag{11}$$

where $\alpha = (1 - 2\epsilon - \epsilon^2)/(1 + \epsilon)^2$ and $f = -(1 + \epsilon) \ln N$.
Again from them one obtains,

$$\overline{\Delta}_f \varphi \geq 2(1 - 2\epsilon - \epsilon^2)\varphi^2, \qquad \text{where} \qquad \varphi = |\nabla f|^2 \tag{12}$$

The coeficient $(1-2\epsilon-\epsilon^2)$ is positive for ϵ in the range $(-1 - \sqrt{2}, -1 + \sqrt{2})$.

This is a crucial technical tool to classify *all static vaccum black holes in* $3+1$.

THEOREM

(R' 2016, to appear) Any static vaccum solutions of the Einstein equations with compact but non necessarily connected horizon is either,

1. *a Schwarzschild solution,*
2. *a Boost,*
3. *is of Myers/Korotkin-Nicolai type, that is, it has the same topology and Kasner asymptotic as the Myers/Korotkin-Nicolai black holes.*

This is a crucial technical tool to classify *all static vaccum black holes in* $3+1$.

Theorem

(R' 2016, to appear) Any static vaccum solutions of the Einstein equations with compact but non necessarily connected horizon is either,

1. *a Schwarzschild solution,*
2. *a Boost,*
3. *is of Myers/Korotkin-Nicolai type, that is, it has the same topology and Kasner asymptotic as the Myers/Korotkin-Nicolai black holes.*

Observe that no assumption is made on the asymptotic a priori like asymptotic flatness. This extends Israel's 1960 uniqueness theorem.

This is a crucial technical tool to classify *all static vaccum black holes in* 3 + 1.

Theorem

(R' 2016, to appear) Any static vaccum solutions of the Einstein equations with compact but non necessarily connected horizon is either,

1. *a Schwarzschild solution,*
2. *a Boost,*
3. *is of Myers/Korotkin-Nicolai type, that is, it has the same topology and Kasner asymptotic as the Myers/Korotkin-Nicolai black holes.*

Observe that no assumption is made on the asymptotic a priori like asymptotic flatness. This extends Israel's 1960 uniqueness theorem.

Myers; PRD; 1987 - *Higher-dimensional black holes in compactified space-times*.
Korotkin, Nicolai; NP B; 1994 - *The Ernst equation on a Riemann surface*.

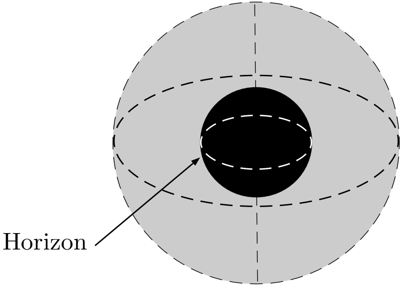

FIGURE: A representation of a Schwarzschild black hole. The manifold M is the region in grey and is diffeomorphic to $\mathbb{R}^3$ minus an open ball (the black ball). The solution is spherically symmetric and thus axisymmetric.

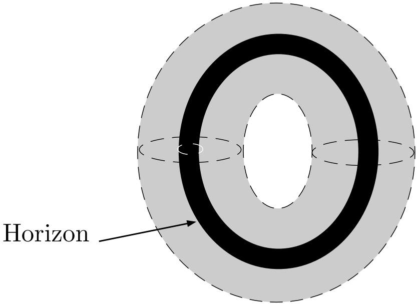

FIGURE: A representation of a Boost black hole. The manifold M is the region in grey and is diffeomorphic to a solid torus minus an open solid torus (the black torus). The solution is axisymmetric.

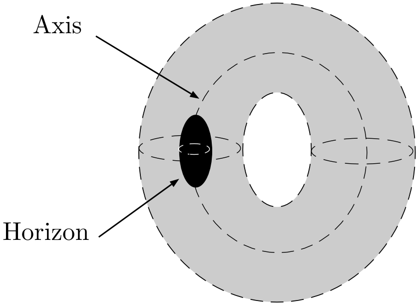

FIGURE: A representation of a Myers/Korotkin-Nicolai black hole. The manifold M is the region in grey and is diffeomorphic to a solid torus minus an open ball (the black ball). The solution is axisymmetric.

Einstein/Klein-Gordon static solutions

The Einstein/Klein-Gordon static equations,

$$Ric + \nabla\nabla f - \nabla f \nabla f = \nabla\phi \circ \nabla\bar{\phi} + \frac{m|\phi|^2}{n-1} g, \tag{13}$$

$$\Delta f - \langle \nabla f, \nabla f \rangle = \frac{m|\phi|^2}{n-1}, \tag{14}$$

$$\Delta\phi - \langle \nabla f, \nabla\phi \rangle = m\phi, \tag{15}$$

where $f = -\ln N$.

Einstein/Klein-Gordon static solutions

The Einstein/Klein-Gordon static equations,

$$Ric + \nabla\nabla f - \nabla f \nabla f = \nabla\phi \circ \nabla\bar{\phi} + \frac{m|\phi|^2}{n-1} g, \tag{13}$$

$$\Delta f - \langle \nabla f, \nabla f \rangle = \frac{m|\phi|^2}{n-1}, \tag{14}$$

$$\Delta\phi - \langle \nabla f, \nabla\phi \rangle = m\phi, \tag{15}$$

where $f = -\ln N$.
After a manipulation one gets,

$$\Delta_f |\nabla\phi|^2 \geq |\nabla\phi|^4 \tag{16}$$

Therefore if M is non-compact and $\partial M = \emptyset$ we get $\phi = const$ and so $\phi = 0$ if $m \neq 0$ by (12).

Einstein/Klein-Gordon static solutions

The Einstein/Klein-Gordon static equations,

$$Ric + \nabla\nabla f - \nabla f \nabla f = \nabla\phi \circ \nabla\bar{\phi} + \frac{m|\phi|^2}{n-1} g, \tag{13}$$

$$\Delta f - \langle \nabla f, \nabla f \rangle = \frac{m|\phi|^2}{n-1}, \tag{14}$$

$$\Delta\phi - \langle \nabla f, \nabla\phi \rangle = m\phi, \tag{15}$$

where $f = -\ln N$.

After a manipulation one gets,

$$\Delta_f |\nabla\phi|^2 \geq |\nabla\phi|^4 \tag{16}$$

Therefore if M is non-compact and $\partial M = \emptyset$ we get $\phi = const$ and so $\phi = 0$ if $m \neq 0$ by (12).

We end up with a vaccum static solution that must be Minkowski or a quotient of it by Anderson's result.

We arrive at,

Theorem

(R', 2015) There are no non-vaccum static geodesically complete solutions of the Einstein/Klein-Gordon equations.

Klein-Gordon fields cannot statically self gravitate without causing a singularity.

AXISYMMETRIC VACCUM STATIONARY SOLUTIONS

To treat this case one needs and extension of the Bakry-Emery context that includes harmonic maps. This was done by,

[Chen, Jost, Hongbing; AGAG; 2012] - *Existence and Lioville Theorems for V-Harmonic maps from complete manifolds*

The technique could be useful to prove existence of rotating stationary solutions like those of Myers/Korotkin-Nicolai.