Scale dependent Einstein-(power)-Maxwell black hole in (2+1) dimensions

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Introduction

The understanding of quantum gravity allows us to get insights in the Black Hole theory and vice-versa.

At low energies, the resulting effective action of gravity shows us a **scale dependence**.

The couplings, which appearing in the effective action, evolve and depend on the scale, i.e. $G_0 \to G_k$ and $e_0 \to e_k$.

Classical Action & EoM

The classical action in three dimensions is

$$I_0[g_{\mu\nu}] = \int d^3x\sqrt{-g}\left[\frac{1}{2\kappa_0}R + \frac{1}{e_0^{2\beta}}\mathcal{L}(F)\right],$$

where $\kappa_0 = 8\pi G_0$ is the Einstein's constant, and $\mathcal{L}(F) = C|F|^\beta$ is the electromagnetic Lagrangian density.

Varying the action respect to the metric field one gets

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{\kappa_0}{e_0^{2\beta}}T_{\mu\nu},$$

where the energy momentum tensor reads

$$T_{\mu\nu} = \mathcal{L}_F g_{\mu\nu} - \mathcal{L}(F)F_{\mu\gamma}F_\nu{}^\gamma.$$

Classical Action & EoM

Note that $\mathcal{L}_F = d\mathcal{L}/dF$ and for static spherically symmetric solutions the electric field is given by

$$F_{\mu\nu} = (\delta^r_\mu \delta^t_\nu - \delta^r_\nu \delta^t_\mu) E(r),$$

while for the metric we make the ansatz

$$ds^2 = -f_0(r)\, dt^2 + f_0(r)^{-1}\, dr^2 + r^2 d\phi^2.$$

Finally, the equation of motion for the Maxwell field $A_\mu(x)$ reads

$$D_\mu \left(\frac{\mathcal{L}_F F^{\mu\nu}}{e_0^{2\beta}} \right) = 0.$$

One only needs to determine the set of functions $\{f(r), E(r)\}$.

Our interest

We will focus on two relevant cases:

1- Einstein Maxwell case ($\beta = 1$)

It is the well known usual electrodynamics.

2- Einstein power Maxwell case ($\beta = 3/4$)

It preserves the traceless property inherited of the usual electrodynamics in (3+1) dimensions.

Classical Solution ($\beta = 1$)

With the solution given by

$$f_0(r) = -M_0G_0 - \frac{1}{2}\frac{Q_0^2}{e_0^2}\ln\left(\frac{r}{\tilde{r}_0}\right), \qquad E_0(r) = \frac{Q_0e_0^2}{r}.$$

The temperature, entropy and specific heat are:

$$T_0 = \frac{1}{4\pi}\left|\frac{Q_0^2}{2e_0^2r_0}\right|, \qquad S_0 = \frac{\mathcal{A}_H(r_0)}{4G_0}, \qquad C_0 = T\frac{\partial S}{\partial T}\bigg|_Q = -S_0.$$

Please, note that the horizon is given by the condition $f_0(r_H) = 0$

$$r_0 = \tilde{r}_0 e^{-\frac{2M_0G_0e_0^2}{Q_0^2}},$$

Classical Solution ($\beta = 3/4$)

With the solution given by

$$f_0(r) = -M_0 G_0 + \frac{4G_0 Q_0^2}{3r}, \qquad E_0(r) = \frac{Q_0}{r^2},$$

The temperature, entropy and specific heat are:

$$T_0 = \frac{1}{4\pi}\left|\frac{M_0 G_0}{r_0}\right|, \qquad S_0 = \frac{\mathcal{A}_H(r_0)}{4G_0}, \qquad C_0 = T\frac{\partial S}{\partial T}\bigg|_Q = -S_0.$$

Please, note that the horizon is given by the condition $f_0(r_H) = 0$

$$r_0 = \frac{4}{3}\frac{Q_0^2}{M_0}.$$

Action with Running Couplings

The gravitational action in three dimensions is

$$\Gamma[g_{\mu\nu}, k] = \int d^3x\sqrt{-g}\left[\frac{1}{2\kappa_k}R + \frac{1}{e_k^{2\beta}}\mathcal{L}(F)\right].$$

Thus, varying with respect to the metric field one gets

$$G_{\mu\nu} = \frac{\kappa_k}{e_k^{2\beta}}T_{\mu\nu}^{\text{effec}},$$

where the **effective** energy-momentum tensor is given by

$$\kappa_k T_{\mu\nu}^{\text{effec}} = \kappa_k T_{\mu\nu}^{\text{EM}} - e_k^{2\beta}\Delta t_{\mu\nu},$$

being the new term:

$$\Delta t_{\mu\nu} = G_k\Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)G_k^{-1}.$$

Action with Running Couplings

In the same way, by varying the action with respect to the scale-field $k(x)$ one obtains the algebraic equations

$$R\frac{\partial}{\partial k}\left(\frac{1}{G_k}\right)=\mathcal{L}(F)\frac{\partial}{\partial k}\left(\frac{16\pi}{e_k^{2\beta}}\right),$$

However, we don't use it! We use the NEC to compute the gravitational coupling.

On the other hand, we take into account that:

$$\mathcal{O}(k)\longrightarrow\mathcal{O}(k(r))\longrightarrow\mathcal{O}(r),$$

and solve with respect to the radial variable.

Solution ($\beta = 1$)

Solving the EoMs one finds

$$G(r) = \frac{G_0}{1+\epsilon r},$$

$$f(r) = -\frac{G_0 M_0}{(r\epsilon+1)^2} - \frac{1}{2}\frac{Q_0^2}{e_0^2(r\epsilon+1)^2}\left(r\epsilon + \ln\left(\frac{r}{\tilde{r}_0}\right)\right),$$

$$e(r)^2 = e_0^2\left[\frac{1+4r\epsilon}{(1+r\epsilon)^3} - \left(4M_0G_0 - 5Q_0^2 + 2Q_0^2\ln\left(\frac{r}{\tilde{r}_0}\right)\right)\frac{r^2\epsilon^2}{(1+r\epsilon)^3}\right],$$

$$E(r) = \frac{Q_0}{r}e_0^2\left[\frac{1+4r\epsilon}{(1+r\epsilon)^3} - \left(4M_0G_0 - 5Q_0^2 + 2Q_0^2\ln\left(\frac{r}{\tilde{r}_0}\right)\right)\frac{r^2\epsilon^2}{(1+r\epsilon)^3}\right].$$

Note that ϵ parametrizes the strength of the scale dependence and when $\epsilon \to 0$ the classical solution is recovered.

Asymptotic space-times

For small radial coordinate the standard singularity, which is present in the classical solution, is maintained

$$R = -f''(r) - \frac{2f'(r)}{r},$$

Which reads

$$R = 2M_0G_0\frac{\epsilon(r\epsilon - 2)}{r(r\epsilon + 1)^4} + \frac{1}{2}\frac{Q_0^2}{e_0^2}\frac{1 + 7r^2\epsilon^2}{r^2(r\epsilon + 1)^4} + \frac{Q_0^2}{e_0^2}\frac{\epsilon(r\epsilon - 2)}{r(r\epsilon + 1)^4}\ln\left(\frac{r}{\tilde{r}_0}\right).$$

Whereas, classically one gets

$$R = \frac{Q_0^2}{2e_0^2r^2}$$

Horizon structure

The zero of the lapse function implies a special expression for radial coordinate, which reads

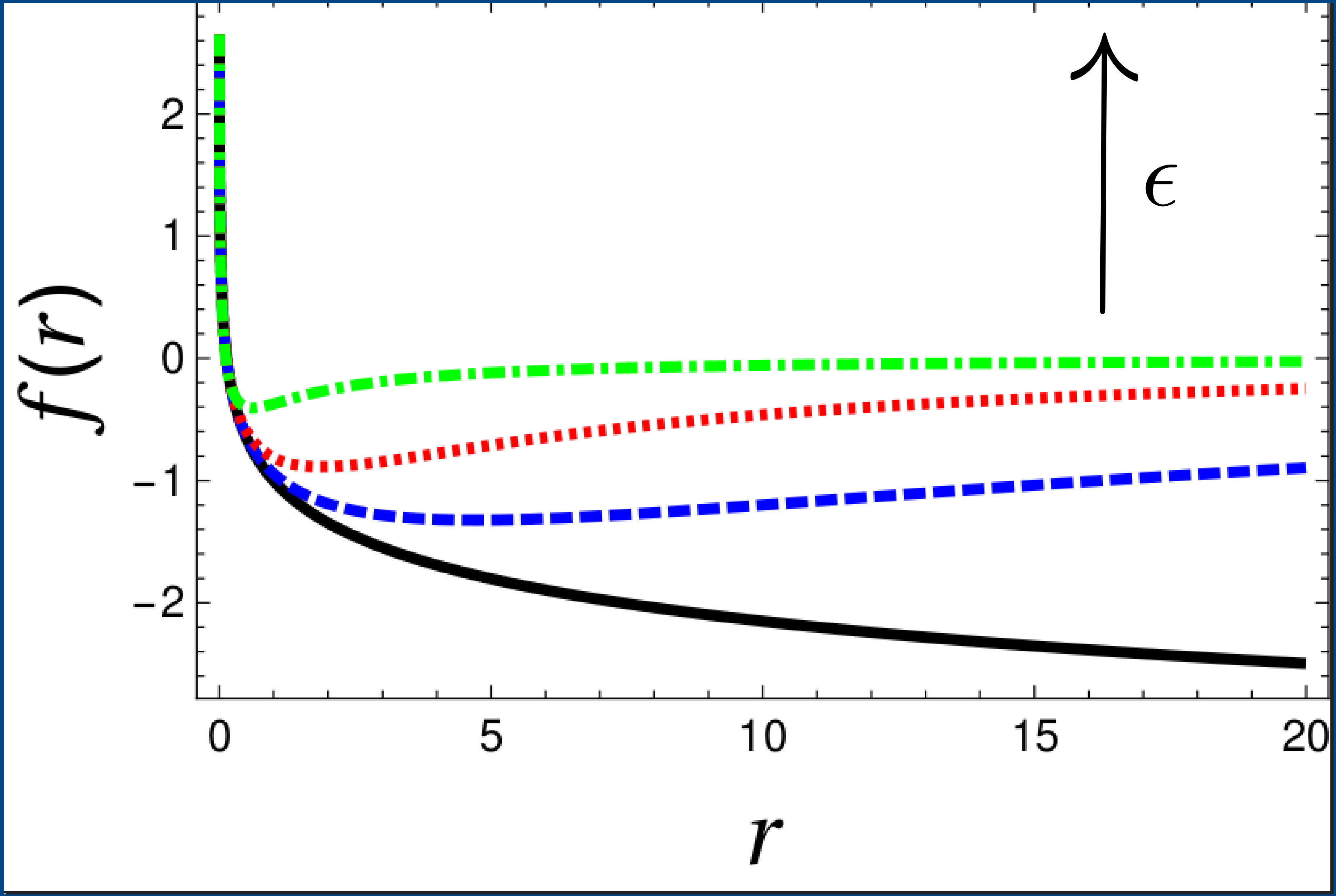
$$r_H = \frac{1}{\epsilon} W \left(\epsilon e^{-\frac{2G_0 M_0 e_0^2}{Q_0^2}} \right),$$

Where $W(\cdots)$ is the so-called Lambert-W function. For small ϵ values the horizon can be expanded according to

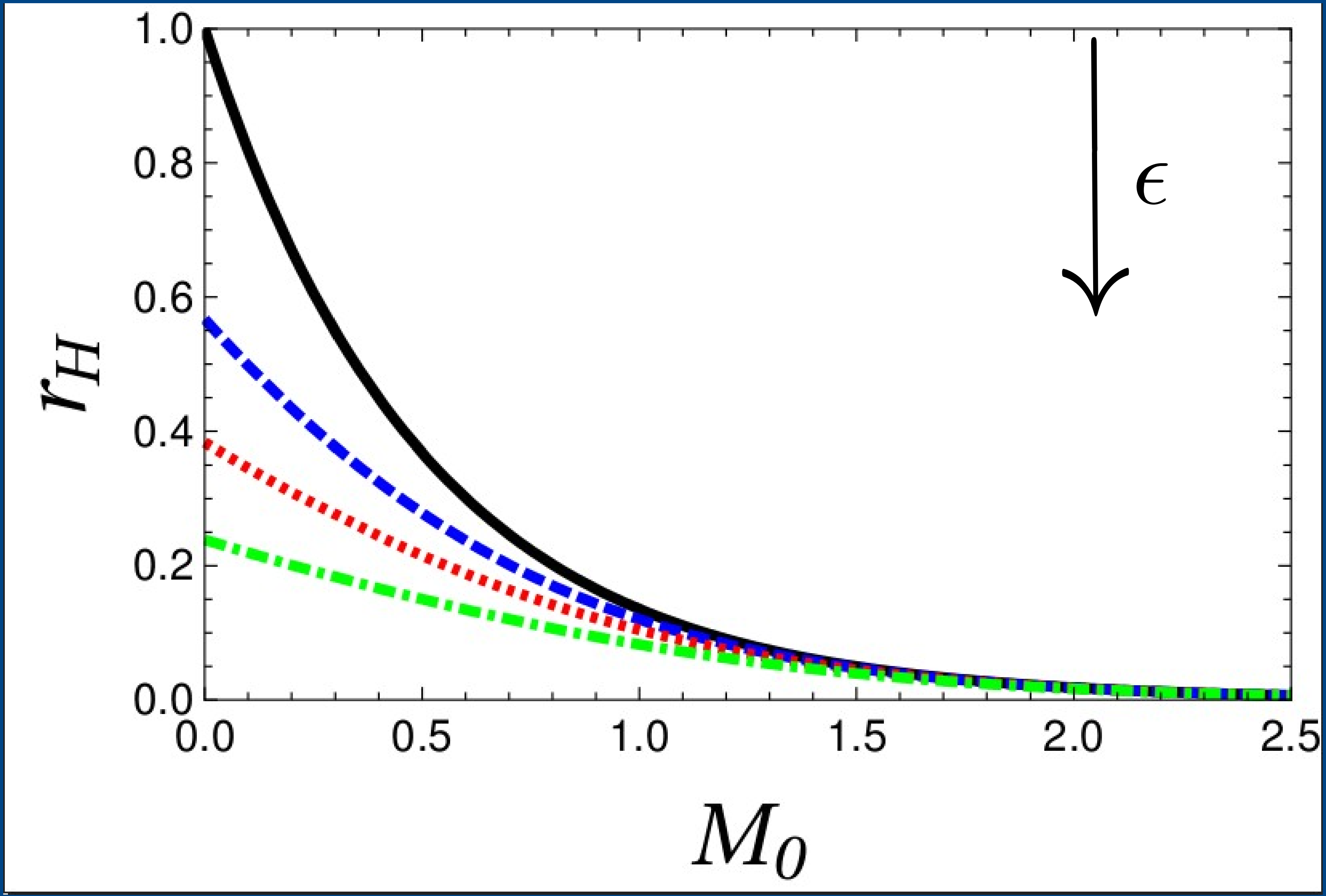
$$r_H \approx r_0 \left[1 - \epsilon r_0 + \mathcal{O}(\epsilon^2) \right],$$

being r_0 the classical horizon

Numerical Results A



Numerical Results B



Black Hole Thermodynamics

We found the black hole temperature, the entropy and the specific heat by using the standard relations:

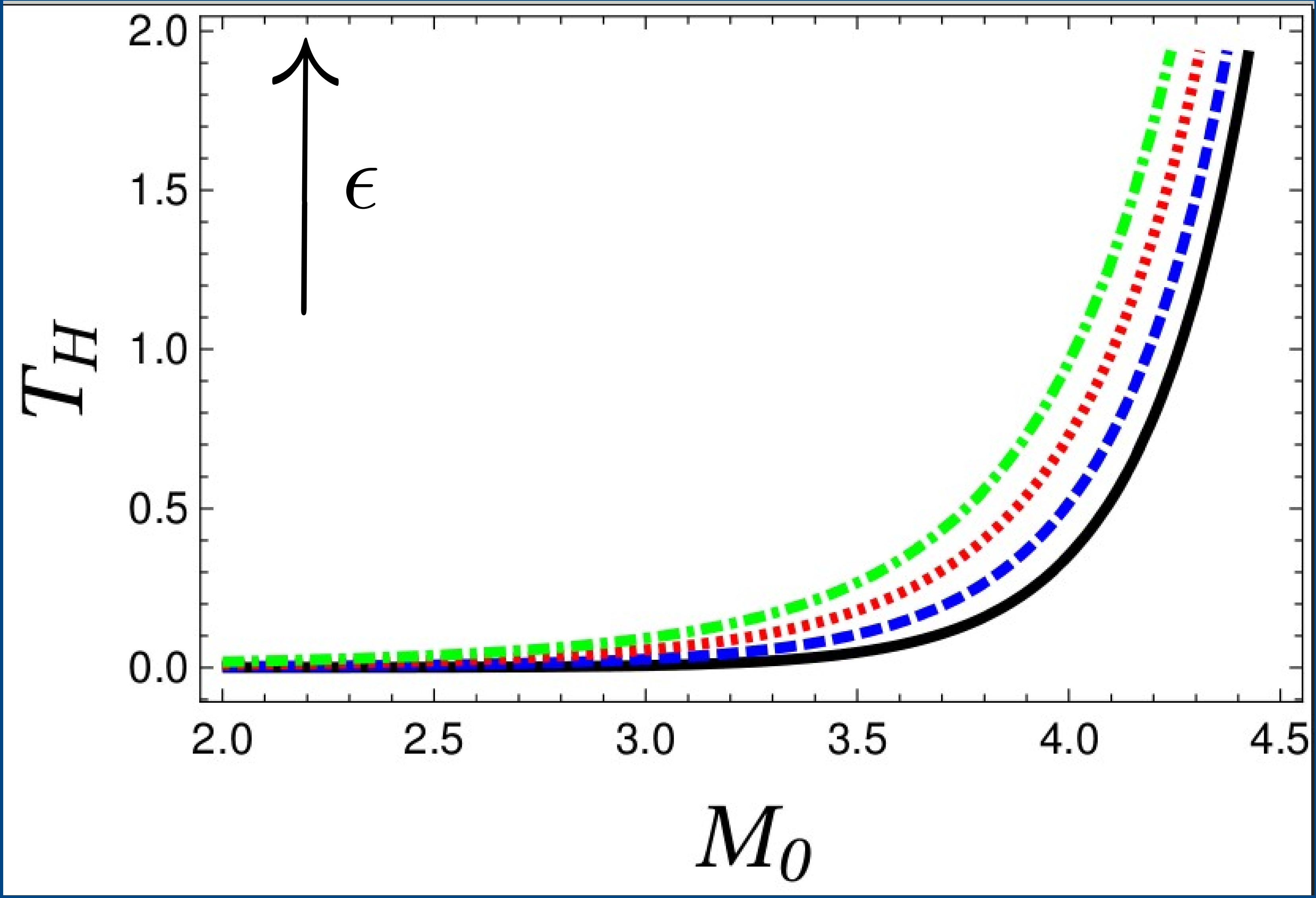
$$T \equiv \frac{1}{4\pi}\left| \lim_{r\to r_H} \frac{\partial_r g_{tt}}{\sqrt{-g_{tt}g_{rr}}}\right| = \frac{1}{4\pi}\left|\frac{Q_0^2}{2e_0^2 r_H(1+\epsilon r_H)}\right|,$$

$$S \equiv \frac{\mathcal{A}_H}{4G(r_H)} = S_0(r_H)(1+\epsilon r_H),$$

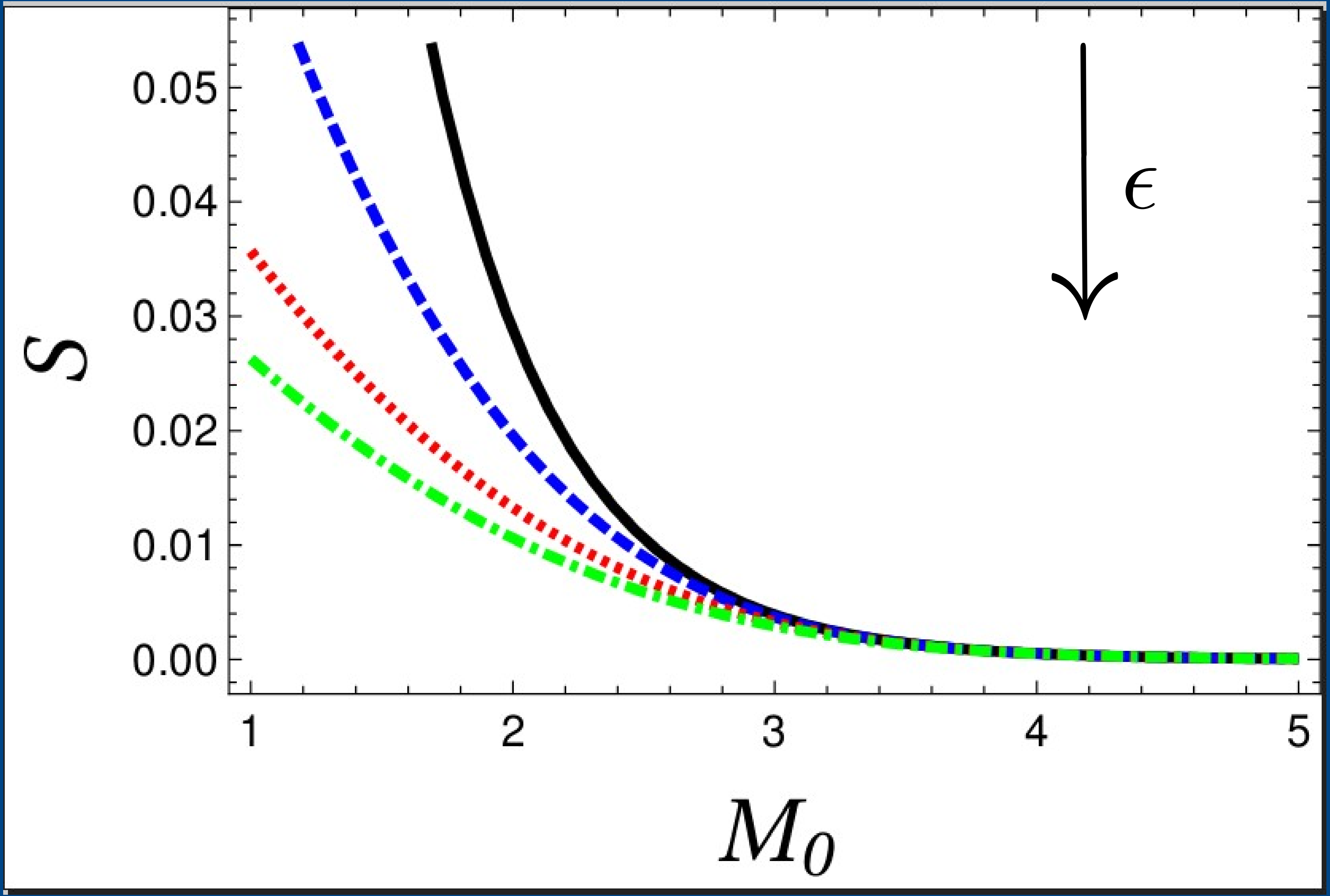
$$C_Q \equiv T\frac{\partial S}{\partial T}\bigg|_Q = -S_0(r_H)(1+\epsilon r_H).$$

Where the scale dependence induces corrections in all quantities.

Numerical Results C



Numerical Results D



Solution ($\beta = 3/4$)

Solving the EoMs one finds

$$G(r) = \frac{G_0}{1+\epsilon r},$$

$$f(r) = \frac{4G_0 Q_0^2}{3r(r\epsilon+1)^3} - M_0 G_0 \frac{\left(r^3\epsilon^2 + 3r^2\epsilon + 3r\right)}{3r(r\epsilon+1)^3},$$

$$e(r)^3 = e_0^3 \left[\frac{2r\epsilon(3r\epsilon+2)+1}{(r\epsilon+1)^4} - \frac{M_0}{4Q_0^2}\frac{r^3\epsilon^2(r\epsilon+4)}{(r\epsilon+1)^4}\right],$$

$$E(r) = \frac{Q_0}{r^2}\left[\frac{2r\epsilon(3r\epsilon+2)+1}{(r\epsilon+1)^4} - \frac{M_0}{4Q_0^2}\frac{r^3\epsilon^2(r\epsilon+4)}{(r\epsilon+1)^4}\right].$$

Note that ϵ parametrizes the strength of the scale dependence and when $\epsilon \to 0$ the classical solution is recovered.

Asymptotic space-times

For small radial coordinate a new singularity, which is not present in the classical solution, is obtained

$$R = -f''(r) - \frac{2f'(r)}{r},$$

Which reads

$$R = -4G_0\epsilon \frac{(M_0 + 4Q^2\epsilon)}{r(r\epsilon + 1)^5}.$$

Whereas, classically one gets

$$R = 0$$

Horizon structure

Applying the condition $f(r_H) = 0$ one obtains the scale dependent horizon which reads

$$r_H = -\frac{1}{\epsilon}\left[1 - \left[1 + 3\epsilon r_0\right]^{1/3}\right],$$

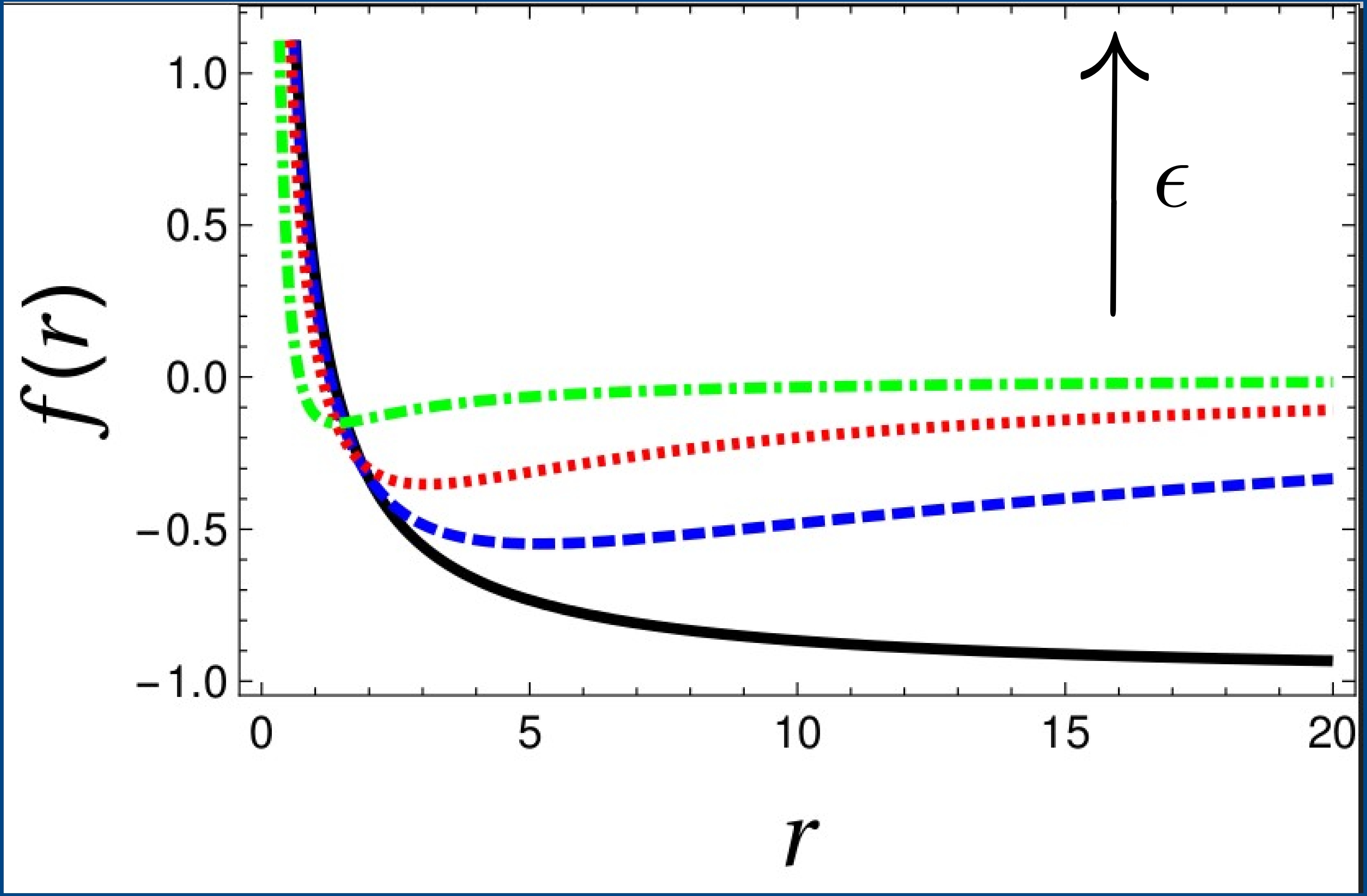
$$r_\pm = -\frac{1}{\epsilon}\left[1 + \frac{1}{2}(1 \pm i\sqrt{3})\left[1 + 3\epsilon r_0\right]^{1/3}\right].$$

For small ϵ values the horizon can be expanded according to

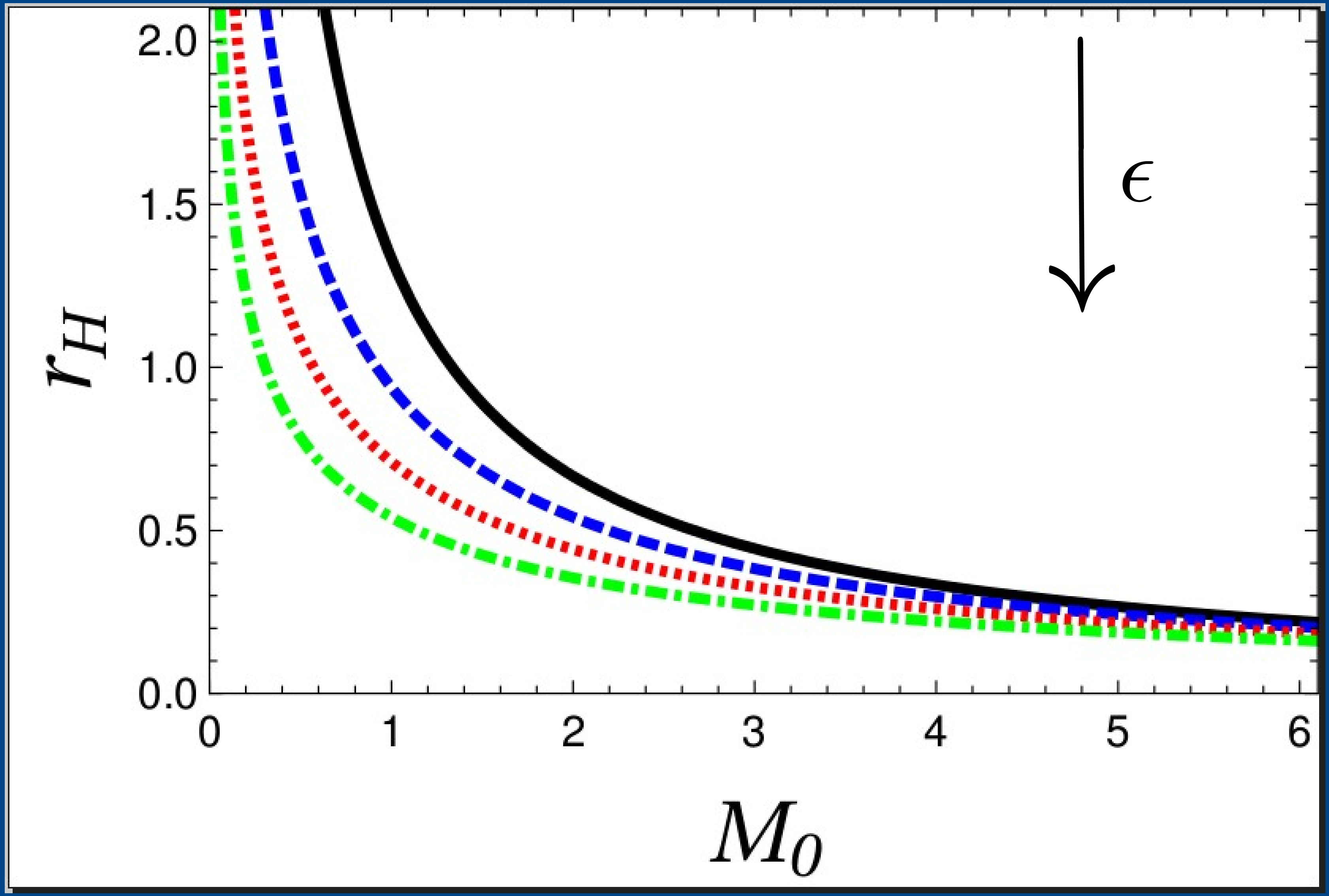
$$r_H \cong r_0\left[1 - \epsilon r_0 + \frac{5}{3}\,(\epsilon r_0)^2 + \cdots\right],$$

Where r_0 is the classical horizon.

Numerical Results E



Numerical Results F



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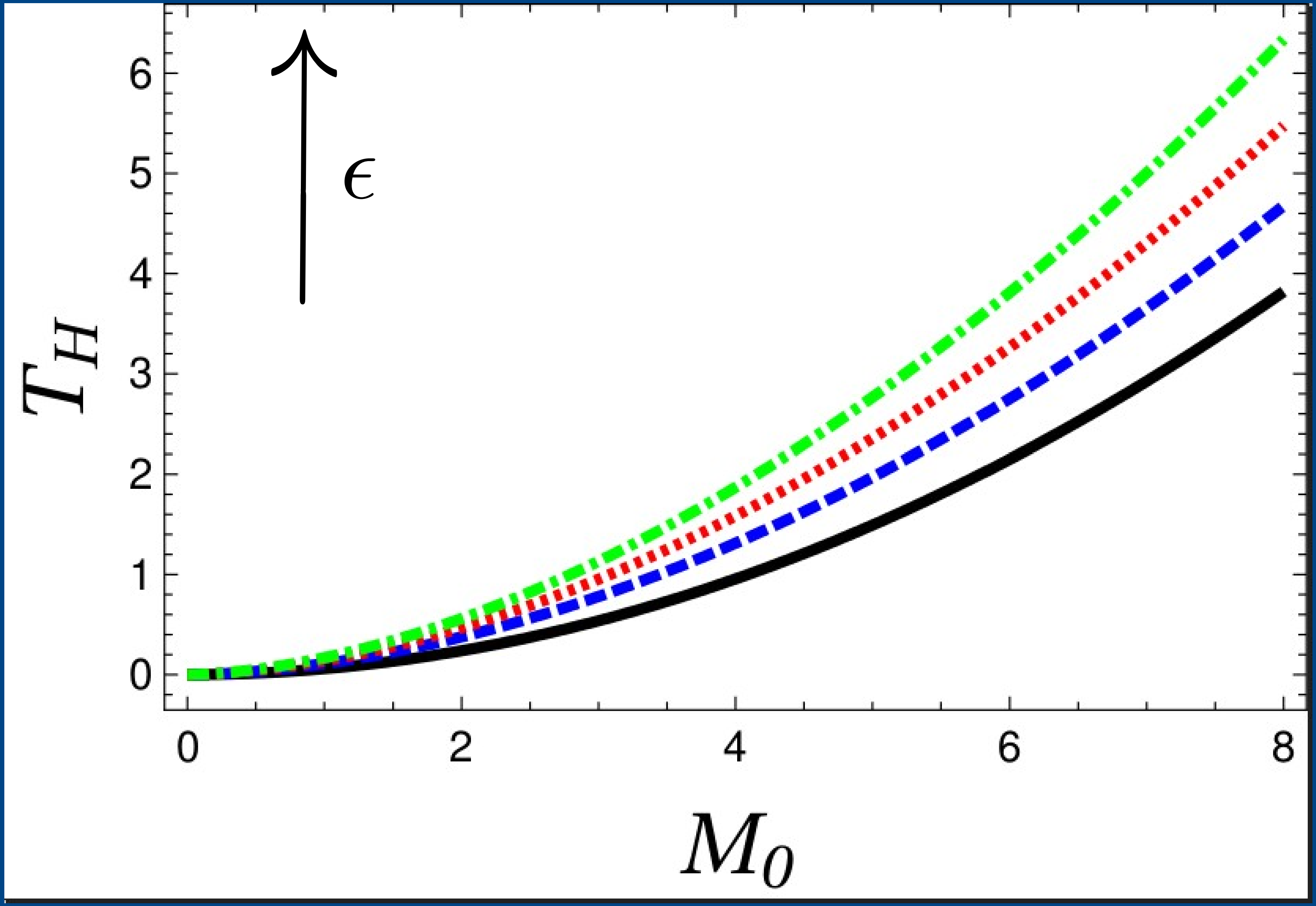
$$S \equiv \frac{\mathcal{A}_H}{4G(r_H)} = S_0(r_H)(1+\epsilon r_H),$$

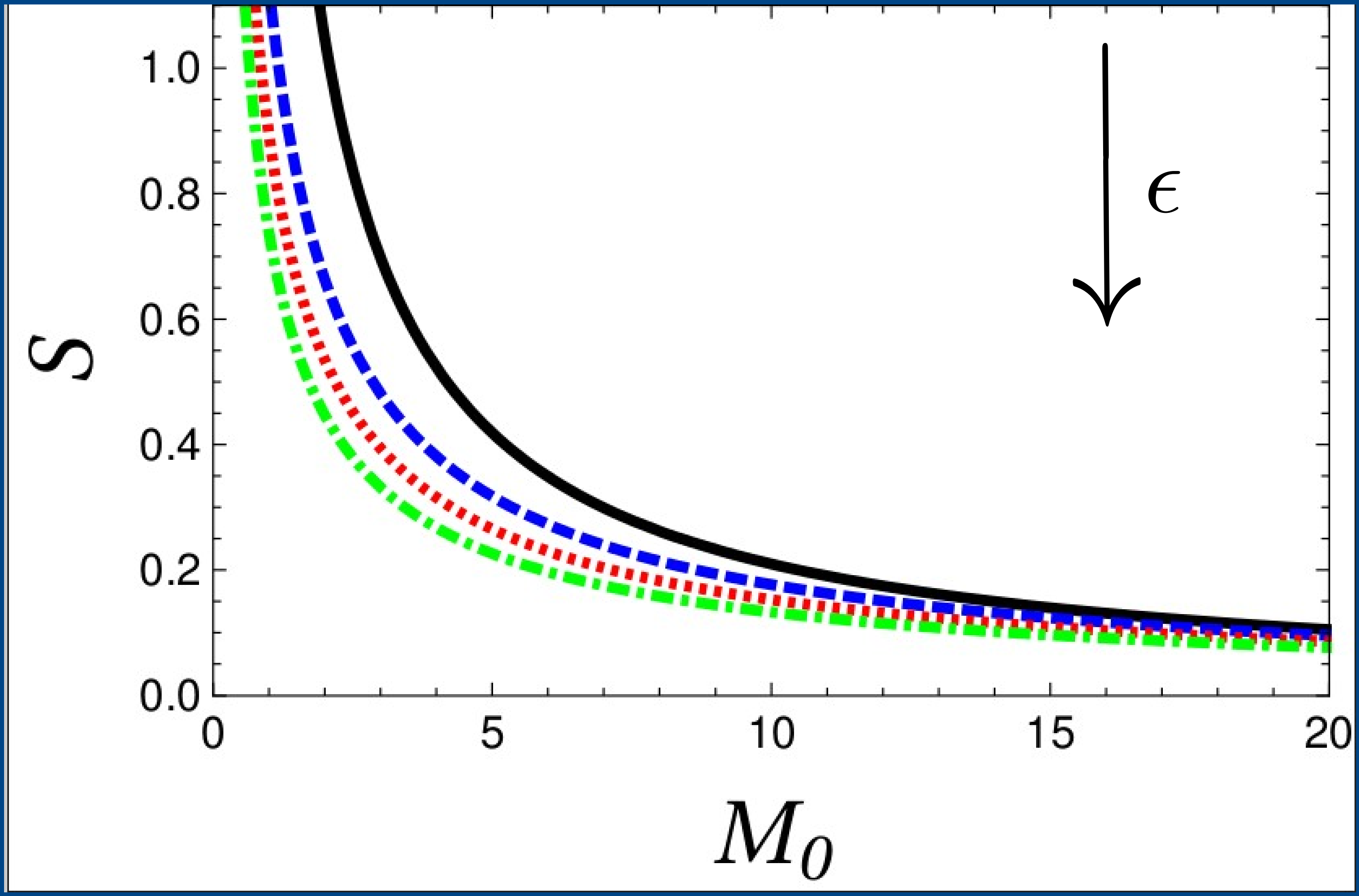
$$C_Q \equiv T\frac{\partial S}{\partial T}\bigg|_Q = -S_0(r_H)(1+\epsilon r_H).$$

Where the scale dependence induces corrections in all quantities.

Numerical Results G



Numerical Results H



Take home messages

1- Scale dependence introduces an extra contribution to the stress energy tensor.

2- Integration constants are crucial in our approach!

3- For small black holes, the usual "area law" holds up to $\mathcal{O}(\epsilon)$, the opposite limit (which occurs when $G(r)$ deviates strongly from G_0) follows an "area × radius law".

4- Due the analytical horizon, it is possible compute exactly the entropy as well as the specific heat.

5- The specific heat indicates that this black hole is unstable.

Extra

NEC with **Running Couplings**

The metric has a general form:

$$ds^2 = -f(r)\,dt^2 + g(r)\,dr^2 + r^2 d\phi^2.$$

Note that, in order to combine $f(r)$ with $g(r)$, we will use the so called "**null energy condition**":

$$T_{\mu\nu} l^\mu l^\nu \stackrel{!}{=} T^m_{\mu\nu} l^\mu l^\nu \geq 0.$$

Which produces:

$$R_{\mu\nu} l^\mu l^\nu = (f \cdot g)' \frac{1}{2rg} = 0,$$

which allows us to rewrite

$$f(r) \cdot g(r) = 1.$$

NEC with Running Couplings

Thus, the running couplings let us get the Newton's function:

$$G(r) = a \left[\int_{r_0}^{r} \sqrt{f(r') \cdot g(r')} \, dr' \right]^{-1},$$

with the general solution:

$$G(r) = \frac{G_0}{1 + \epsilon r}$$

Note that the functional form of $G(r)$ is a consequence of the null energy condition!