Entanglement entropy and mutual information of smeared field observables

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Motivation

Entanglement entropy is of key importance for semiclassical and quantum gravity

…A) On its own right.

- Measure of correlations of quantum degrees of freedom on a classical background.
- One-loop contribution to Bekenstein-Hawking entropy.
- Crucial for the firewall argument and other black hole evaporation puzzles.

…B) As holographic counterpart of area.

- Fundamental to attempts to construct spacetime from quantum informational notions.

But entanglement entropy of quantum fields is not well-defined!

Also: Actual measurements never probe the continuum of field degrees of freedom.

Example

How entangled is the field in region A with the field in region B?

This and similar questions arise naturally in the many applications of entanglement entropy.

Using the exact “geometric” entanglement entropy that involves **all** the d.o.f. in a region, they are difficult to answer.

But suppose we ask a question more directly related with actual measurements:

How entangled is the field measured by an extended detector in A, with the one in B?

Smeared field observable subalgebra

Proposal: Focus on the entanglement entropy (and other informational properties) of a **discrete set of observables** built from a quantum field.

We will consider a **subalgebra** of field and field momentum observables:

$$A = (\phi_1, \ldots, \phi_N, \Pi_1, \ldots, \Pi_N)$$

They are defined by integrating the field and field momentum against **smearing functions**.

$$\phi_i = \int d^3x \; \phi(\vec{x}) f_i(\vec{x})$$

$$\Pi_i = \int d^3x \; \Pi(\vec{x}) \tilde{f}_i(\vec{x})$$

Integration is over spacelike hypersurface.

(For the moment assume Minkowski for simplicity.)

For example we can have (ϕ_A, ϕ_B, Π_A, Π_B) integrating ϕ and Π against smearing functions peaked sharply in regions A and B, representing detector-measured averages.

Entropy of subalgebra of smeared field observables

Symplectic form on this subalgebra:

$$[\phi_i, \Pi_j] = i \int d^3x \; f_i(\vec{x}) \, \tilde{f}_j(\vec{x}) \;\; = i \, \Omega_{ij}$$

$$\Omega = \begin{pmatrix} 0 & \Omega_{11} & 0 & \Omega_{12} & \cdots \\ -\Omega_{11} & 0 & \Omega_{21} & 0 & \\ \cdots & & & & \end{pmatrix}$$

$$[\phi_i, \phi_j] = 0 = [\Pi_i, \Pi_j]$$

state-independent

Correlations matrix:

$$G = 2 \begin{pmatrix} \langle (\phi_1)^2 \rangle & \frac{1}{2}\langle \phi_1 \Pi_1 + \Pi_1 \phi_1 \rangle & \langle \phi_1 \phi_2 \rangle & \cdots \\ \frac{1}{2}\langle \phi_1 \Pi_1 + \Pi_1 \phi_1 \rangle & \langle (\Pi_1)^2 \rangle & \frac{1}{2}\langle \phi_2 \Pi_1 + \Pi_1 \phi_2 \rangle & \\ \cdots & & & \end{pmatrix}$$

state-dependent (state assumed Gaussian)

Entanglement entropy:

$$S_A = \sum_j \left[\left(\frac{\nu_j + 1}{2} \right) \ln \left(\frac{\nu_j + 1}{2} \right) - \left(\frac{\nu_j - 1}{2} \right) \ln \left(\frac{\nu_j - 1}{2} \right) \right]$$

where $\pm \nu_j = \mathrm{Eigenvalues}[iJ]$ $\qquad J = \Omega^{-1} \cdot G$

Interpretation

S_A measures the entanglement between the field degrees of freedom captured in the subalgebra A, and those in its orthogonal complement.

Same expression that works on a discrete lattice of oscillators. (Bianchi, Hackl and Yokomizo 2015)

S_A is finite and well-defined for any finite-dimensional subalgebra.

We can define the mutual information I_{AB} between two subalgebras:

$$I_{AB} = S_A + S_B - S_{AB}$$

Measures the entanglement between the degrees of freedom captured in A and those captured in B.

Example 1: single observable with Gaussian smearing

$$\phi_0 = \frac{1}{(2\pi)^{3/2} R^3} \int d^3x \, \mathrm{e}^{-r^2/2R^2} \, \phi(\vec{x})$$

$$\Pi_0 = \frac{1}{(2\pi)^{3/2} R^3} \int d^3x \, \mathrm{e}^{-r^2/2R^2} \, \Pi(\vec{x})$$

Field and field momentum measured by a detector with Gaussian profile with size R at origin.

Work with massless scalar for simplicity.

Symplectic structure: $\Omega_0 = \frac{1}{i} [\phi_0, \Pi_0] = \frac{1}{8\pi^{3/2} R^3}$

Correlators in the Minkowski vacuum:

$$\langle (\phi_0)^2 \rangle = \frac{1}{8\pi^2 R^2} \qquad \langle \{\phi_0, \Pi_0\} \rangle = 0$$

$$\langle (\Pi_0)^2 \rangle = \frac{1}{8\pi^2 R^4}$$

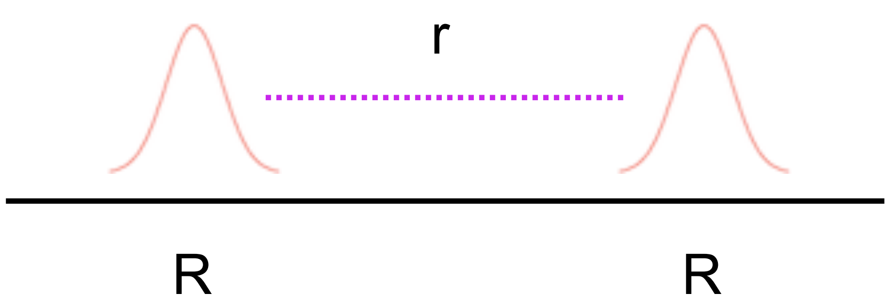
Entanglement entropy for this subalgebra:

$$S(\nu) = \left(\frac{\nu+1}{2}\right) \ln \left(\frac{\nu+1}{2}\right) - \left(\frac{\nu-1}{2}\right) \ln \left(\frac{\nu-1}{2}\right)$$

with $\nu = \frac{2}{\sqrt{\pi}}$

$S \approx 0.24$ (independent of R)

Example 2a: two observables with Gaussian smearing

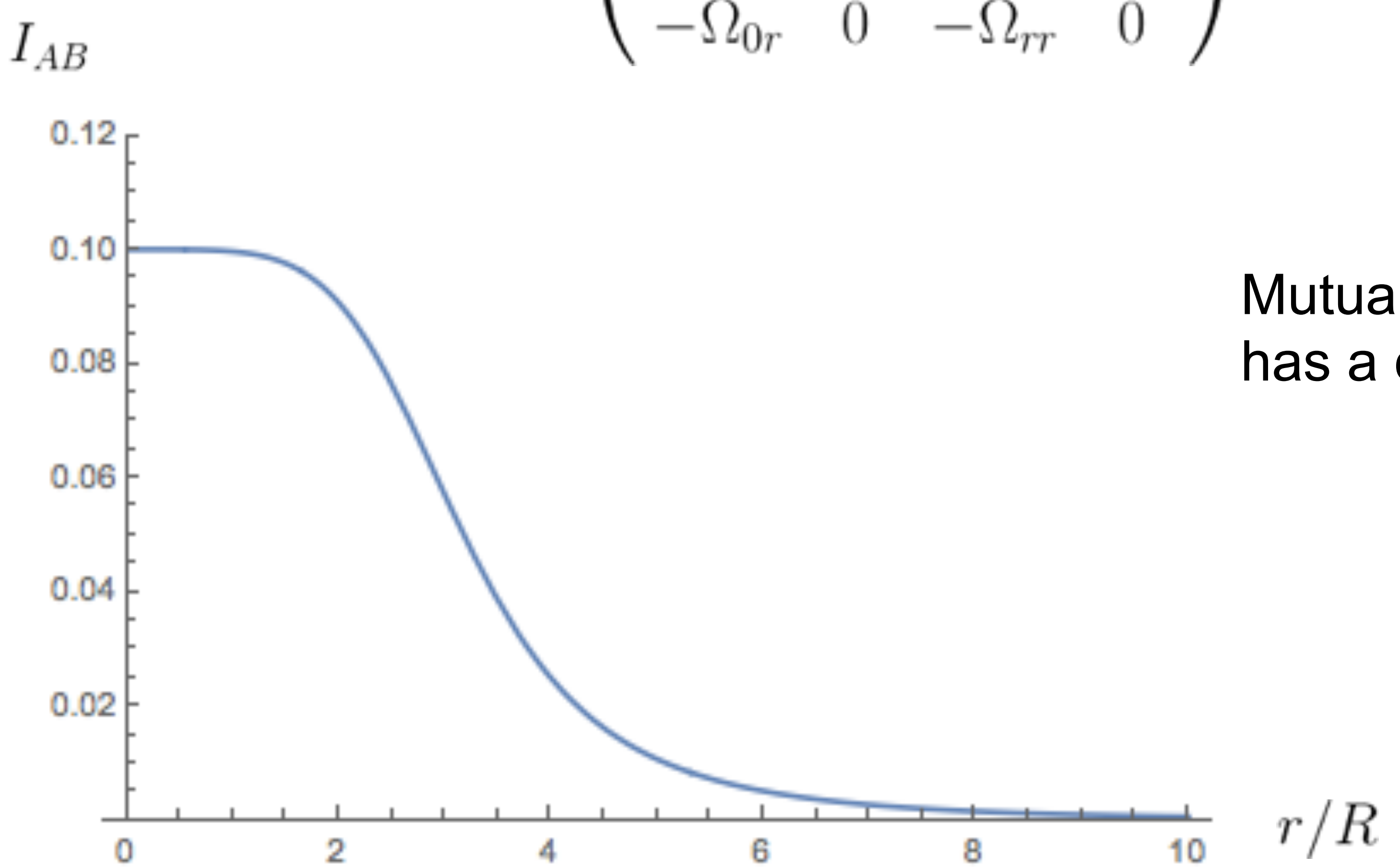


$$S_A = S_B \approx 0.24$$

$$S_{AB} = \mathrm{Tr}\left(\frac{1+iJ}{2}\right)\ln\left|\frac{1+iJ}{2}\right|$$

computed from:

$$\Omega_{AB} = 2\begin{pmatrix} 0 & \Omega_{00} & 0 & \Omega_{0r} \\ -\Omega_{00} & 0 & -\Omega_{0r} & 0 \\ 0 & \Omega_{0r} & 0 & \Omega_{rr} \\ -\Omega_{0r} & 0 & -\Omega_{rr} & 0 \end{pmatrix} \qquad G_{AB} = 2\begin{pmatrix} \langle(\phi_0)^2\rangle & 0 & \langle\phi_0\phi_r\rangle & 0 \\ 0 & \langle(\Pi_0)^2\rangle & 0 & \langle\Pi_0\Pi_r\rangle \\ \langle\phi_0\phi_r\rangle & 0 & \langle(\phi_r)^2\rangle & 0 \\ 0 & \langle\Pi_0\Pi_r\rangle & 0 & \langle(\Pi_r)^2\rangle \end{pmatrix}$$



Mutual information between both observables has a characteristic falloff in Minkowski vacuum:

$$I_{AB} \sim \left(\frac{R}{r}\right)^4$$

Example 2b: two Gaussian observables in cosmology

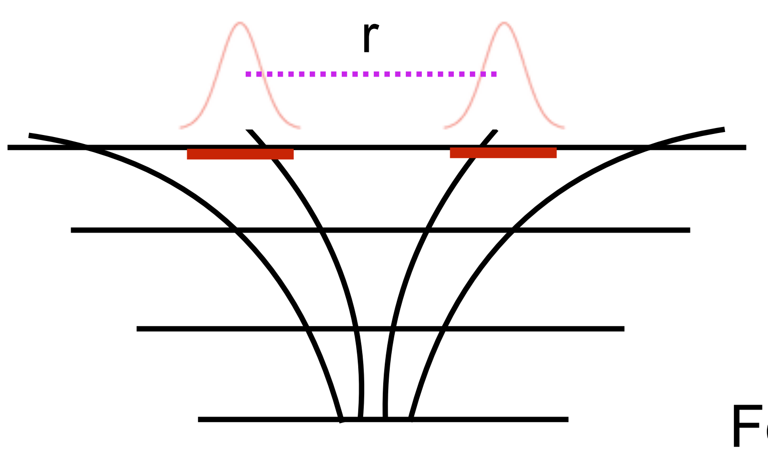
Massless field on spatially flat FLRW background.

At a fixed time t, define observables:

$$\phi_0 = \frac{a^3(t)}{(2\pi)^{3/2}R^3}\int d^3x\, \mathrm{e}^{-|\vec{x}|a^2(t)^2/2R^2}\,\phi(\vec{x}) \qquad \phi_r = \frac{a^3(t)}{(2\pi)^{3/2}R^3}\int d^3x\, \mathrm{e}^{-|\vec{x}-\vec{x}_r|a^2(t)^2/2R^2}\,\phi(\vec{x})$$

simil. for Π_0, Π_r.

$$r = a\,|\vec{x}_r|$$



Observables at physical distance r, representing field averages over regions of physical size R.

For any given choice of background a(t) and quantum state, we can compute the mutual information between the subalgebras associated to the two observables and track its time dependence.

Example 2b: Near-singularity limit

Toy model: $a(t) \propto t^{\alpha}\,, \qquad 0 < \alpha < 1$

Consider a general homogeneous and isotropic Hadamard state. Focus on regime:

$$R \ll t \ll r$$

“Pointlike” observables averaging field over very small region (no overlap), studied in limit where time from singularity much smaller (= curvature scale larger) than distance between observables.

$$I_{AB} \sim C \left(\frac{R}{r}\right)^4 \left(\frac{t}{r}\right)^{2-4\nu}$$

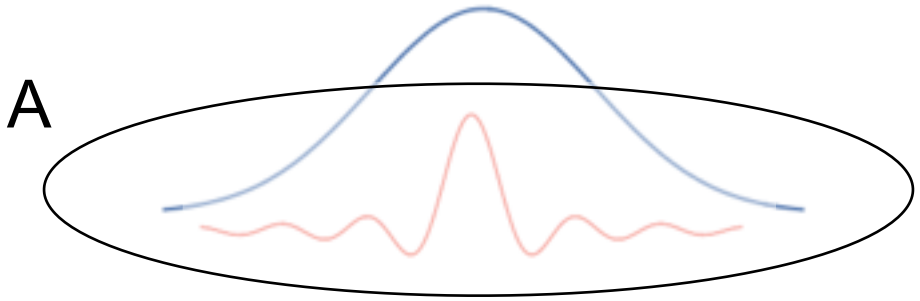
C state-dependent number.

$\nu = |\alpha - 1/2| \longrightarrow 2 - 4\nu > 0$

Mutual information between distant points at fixed separation vanishes as singularity is approached.

Geometric entropy limit

What is the relation between the entanglement entropy of a subalgebra and the geometric entropy of a region?



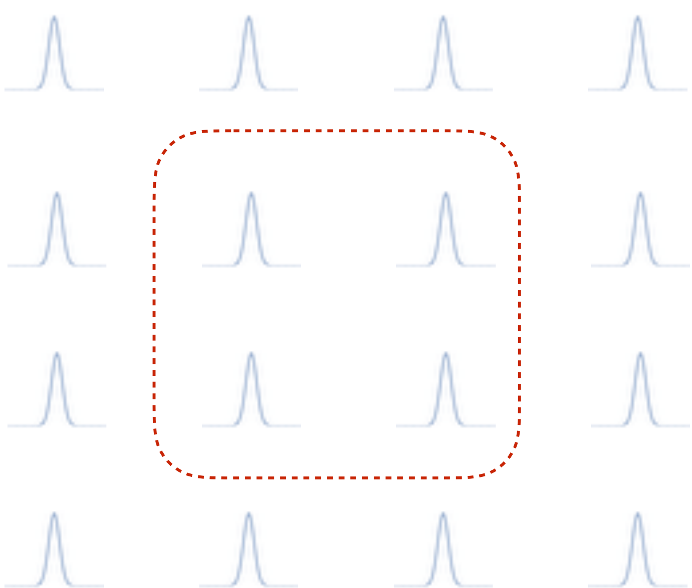
Take a basis of modes expanding the field in a region (that vanish outside it) and use them as smearing functions to define our observables.

The limit in which all this basis is used corresponds to the (divergent) geometric entropy.

We can do this construction for a spherical region in Minkowski space, and recover the area law.

Area law in lattice of pointlike observables

There is a simpler way to recover the area law from our construction that is not a formal geometric entropy limit.



Define a lattice of Gaussian observables with very small size R compared to the separation between them.

(The positions can be regular as depicted, or random sampled).

Not a lattice field theory!

Lattice of observables in a continuum QFT.

The entropy of the subalgebra contained in a region does not scale as the area of the region.

(It includes correlations with degrees of freedom in the region, as well as out.)

But the **mutual information** between the observables inside the region and those outside **scales as the area**.

Summary and outlook

- Entanglement entropy and mutual information defined for subalgebras of smeared field observables probing finite number of degrees of freedom.

- Advantages over conventional geometric entropy of a region:
  - Finite and well-defined.
  - Closer connection with experiments.
  - Easy to compute.
  - Allows us to probe questions of entanglement in a wide range of settings.

- Example: mutual information between two spatially separated Gaussian observables.
  - $\sim 1/r^4$ in Minkowski.
  - Goes to zero a approaching the singularity at fixed r in FLRW toy model.

- Mutual information between observables internal and external to a region satisfies area law.

- Potential additional applications to cosmology, black holes, holography…